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THE UNIVERSITY OF ALBERTA  
ANALYSIS OF CLASS-TEACHER TIMETABLE PROBLEMS

by



GEORGE ARON NEUFELD

A THESIS  
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THE UNIVERSITY OF ALBERTA  
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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled " ANALYSIS OF CLASS-TEACHER TIMETABLE PROBLEMS " submitted by GEORGE ARON NEUFELD in partial fulfilment of the requirements for the degree of Doctor of Philosophy.



## ABSTRACT

The purpose of this study is to address some problems of resource allocation which are exemplified by the class-teacher timetable problem. A fundamental model within class-teacher timetable problems is the coloration of graphs. A general method is developed for the determination of the existence of an  $n$ -coloration of a graph. A condition is identified that is sufficient for the order of computation required to be bounded by a polynomial. The condition is generalized to include cases when the order of computation is moderately exponential. The method is applied to some related graph theoretic problems in order to show the significance of the sufficiency condition.

Two original theorems are proved which pertain to graphs with vertex constraints. Specifically, these constraints are the vertices which are preassigned to specific colors, and vertices which are not to be assigned to given colors. The results show that graphs with these extraneous constraints reduce to graphs without such constraints. Hence, the existence of an  $n$ -coloration of a graph with these constraints may be established by using known strategies. The first of the two theorems has



two further implications. First, it provides the basis for a general and flexible method for determination of an  $n$ -coloration. Hence, it provides the basis for a method for determination of a solution to class-teacher timetable problems. Second, it provides a previously unknown necessary and sufficient condition for the existence of a solution to a particular class-teacher timetable problem, which has been reported in the literature.



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## CHAPTER 1

### INTRODUCTION

This thesis is concerned with a resource allocation problem that many learning institutions have considered since the advent of the computer: The class-teacher timetable problem is defined to contain,

a set of teachers  $T = \{t_i\}$      $i = 1, \dots, \alpha$ ;

a set of classes  $C = \{c_j\}$      $j = 1, \dots, \beta$ ; and

a set of hours  $H = \{h_k\}$      $k = 1, \dots, \gamma$ .

The elements of sets  $T$ ,  $C$ , and  $H$  are to be matched so as to meet specified requirements (constraints). Such a matching, if it exists, will be called a solution. The class-teacher timetable problem will be referred to as the CTT problem.

Underlying all the apparent complexities of CTT problems are some well defined fundamental problems which for the most part are unsolved and which are considered to be difficult by many proficient researchers. The purpose of this study is to address some of these problems.

In Chapter 2, which provides the background to the present research, it is shown that the coloration of a graph is equivalent to a fundamental problem within CTT problems. Within this equivalence, classes correspond to vertices, and hours correspond to colors. A



graph model can represent all constraints which prevent or require the assignment of specified classes to the same hour. This includes those constraints that prevent a teacher from having to meet more than one class in any given hour. Considerable attention has been given to the minimum coloration of a graph. However, since this problem has not yet been solved generally, there are cases when the very existence of a coloration remains in question.

In Chapter 3, a method for the determination of the existence of an  $n$ -coloration of a graph is given. A condition that is sufficient for the computation required to be bounded by a polynomial is identified and generalized to include cases when the order of computation is moderately exponential. A comparison of the results to previously known methods is made. To enhance the significance of the sufficiency condition, it is applied to graph theory.

When other constraints occurring in CTT problems are imposed upon the graph model, the question of the existence of a coloration (solution) has remained unanswered. In Chapter 4, two types of constraints occurring in practical CTT problems are considered: Those where a certain class must meet on a given hour, and those where a certain class cannot meet on given hours. Theorems are proven which reduce graphs to a form that contains no such 'extraneous' constraints. The immediate significance of the theorems is given.



Several methods have been developed for the determination of the minimal coloration of a graph; or equivalently, for the determination of the existence of a coloration with a given number of colors. Most of these methods give a solution to the corresponding CTT problem. In practice, however, these provide a rather inflexible means of meeting other desirable requirements of the problem. In Chapter 4, it is shown how the results of an original theorem proved in this research may be applied so as to provide more flexible methods for obtaining a solution.

Chapter 5 illustrates the results of the earlier chapters, with an example of a CTT problem. The conclusions are stated regarding CTT problems for which previously known methods were unable to determine the existence of a solution. The conclusions pertain to both constraints for which previous work did apply and constraints which until now had remained unanswered. A general method for determining a solution is given. It is shown that a necessary and sufficient condition for the existence of a solution to CTT problems, considered by Gotlieb, is immediate from the results contained within this report. Lastly, the results are stated which relate to graph theory.



## CHAPTER 2

### EXAMINATION OF APPLICABLE APPROACHES

#### 2.1 Background

The approaches that have been applied, or might be considered applicable, to the CTT problem are summarized in the following subsections. Pertinent citations are presented as well as a criticism of each technique. Conclusions derived from an investigation of the various approaches are presented in Section 2.2. Finally, four main objectives of this research are formulated and are related to the class-teacher timetable problem.

##### 2.1.1 Heuristic Approaches

Let the CTT problem be posed as a  $\beta$ -stage decision problem where at each stage  $j$ , a decision pertaining to the assignment of class  $c_j$  to some hour  $h_k$  must be made. The decision at stage  $j$  must not violate any of the constraints in the problem formulation pertaining only to class  $c_j$ , as well as those relating  $c_j$  to classes  $c_1, \dots, c_{j-1}$ . The latter of the above constraints would include those that prevent any teacher from having to meet more than one of his classes in any given hour. Also the decision must, if possible, be made so that it is possible to make a decision at each of stages  $j+1, \dots, \beta$ . At any stage, no decision is attainable if every possible assignment violates some constraint(s).



Heuristic approaches [2,3,6,10,12,17,23,28,31,39,41,44,47] have given rise to criteria such as the following, which are used as strategies for finding a solution.

1. The order in which the classes are assigned to specific hours are arranged, so as to increase the odds of being able to make a decision at each of the  $\beta$  stages. For example, classes considered more difficult to schedule are assigned to specific hours first [41,47].

2. If there is any choice at each stage  $j$ , class  $c_j$  is assigned to an hour so as to increase the chances of being able to make a decision at each of the remaining stages. To illustrate, a class may be assigned to the hour already having the most classes assigned to it [28]; thereby, the use of unassigned hours is minimized.

None of these are deterministic in that an impossible decision at some future stage is always prevented.

If at some stage  $j$  no decision is possible, the alternatives used are:

1. Some (all) previous decision(s)  $r$ ,  $r < j$ , are altered in an attempt to make a decision at stage  $j$  possible.



2. The classes are reordered and the entire process is repeated, beginning at stage 1 (see [39]).

3. The class is assigned, so as to minimally violate the constraints relating class  $c_j$  to classes  $c_1, \dots, c_{j-1}$ . This is equivalent to removing constraints so that a decision at stage  $j$  is possible.

In practice, only restricted forms of alternative 1 can be used since the order of computation required can approach that equal to total enumeration of all feasible and nonfeasible solutions. An algorithm using alternative 2 does not necessarily terminate. Alternative 3 would be acceptable provided it were known that a solution to the original problem does not exist.

GASP[31], a system based on heuristic techniques, has been developed to generate solutions for CTT problems. Many users have found GASP to be unacceptable because it was unable to attain solutions to real problems. More recently, IBM, who supported the development of GASP, announced a new system called SOCRATES. It uses the same concepts as the University of Waterloo system [45] which generates reports, thus making the adjustment of an existing timetable easier.



It is of interest to observe that the difficulties in heuristic approaches are the same as those of branch and bound approaches to the CTT problem.

### 2.1.2 Statistical Sampling Techniques

Sherman [43] (also see [42]) used statistical sampling to find solutions to the combinatorial problems corresponding to CTT problems. It seems likely that if the problem has many constraints, then the probability of finding a solution by sampling becomes relatively small. Similar observations on this approach are reported in [22].

The method of Formby [20] is a variation of statistical sampling to obtain a minimum coloration of a graph, a problem that will be discussed below.

### 2.1.3 Combinatorial Approaches

Gotlieb [21] gives a condition for a feasible solution to exist for the following CTT problem: The problem is defined by

a set of teachers  $T = \{t_i\} \quad i=1, \dots, \alpha,$

a set of classes  $C = \{c_j\} \quad j=1, \dots, \beta,$

a set of hours  $H = \{h_k\} \quad k=1, \dots, \gamma,$

and an initial requirements matrix  $R$  with elements  $r_{ij}$ . The  $r_{ij}$  are non-negative integers representing the number of hours  $t_i$  is to meet



$c_j$ , in the interval for which the timetable is being constructed. Each class  $c_j$  refers to a group of students. The problem statement corresponds to real problems that often occur in public schools. Gotlieb points out that Hall's algorithm [25] provides a method for finding a solution. Based on a necessary condition, Csima and Gotlieb [13] describe a basic iteration for constructing a schedule for the above problem with preassignments. Csima and Gotlieb conclude with:

'None of the references mentioned consider problems which can be interpreted as constructing a time-table with preassignments, nor has it been possible to develop a proof from these references that the basic iteration described above will always provide a solution when one exists. On the other hand, as discussed in the previous section, every case tried so far has been successful.'

Winters [48], Lions [32, 4, 6] and Dempster [15,16] have investigated these various approaches to finding a solution. Lions [33] has reported a counter-example to this hypothesis of Csima and Gotlieb.

Gotliebs's statement of the CTT problem assumes that all constraints, except those expressed in the matrix  $R$ , can and must be stated in the form of preassignments. This itself can be a major task. As an illustration, suppose there are many constraints preventing pairs of classes from being offered at the same time; that is, students having to enroll in both so as to meet academic program requirements.



Another combinatorial type approach for finding solutions to 'scheduling' problems was given by Turksen and Holzman [44]. Let  $X = [x_{jk}]$  be a  $\beta \times \gamma$  solution matrix where  $x_{jk} = 1$  if class  $c_j$  is assigned to hour  $h_k$ . Any solution, feasible or nonfeasible, of the CTT problem can be represented by the  $\beta \cdot \gamma$  boolean vector  $x_{\beta\gamma} \dots x_{\beta 1} \dots x_{1\gamma} \dots x_{11}$  in the  $2^{\beta\gamma}$ -space of boolean lattice points. Corresponding to each such point is a unique positive integer called a designation number. Also, corresponding to each pair of classes  $c_j$  and  $c_{j'}$  that are not to be assigned to the same hour there are  $\gamma$  logical constraints  $x_{jk}x_{j'k} = 0, k = 1, \dots, \gamma$ . Turksen and Holzman determine the nonfeasible points corresponding to a logical constraint in terms of the designation numbers, as opposed to the method of truth tables as suggested by Akers and Friedman [1]. The set of all nonfeasible points is realized by set operations on the sets of designation numbers corresponding to each of the logical constraints. However, their result does not provide a feasible method for the determination of the existence of a solution; since, the order of computation compares to that of total enumeration. The order of computation refers to the number of basic steps required to accomplish the desired result.

#### 2.1.4 Graph Theoretic Approaches

A graph  $G$  consists of the finite nonempty set  $V$  of vertices with  $|V| = m$  and a set  $E$  of edges. Each edge joins two distinct vertices.



Two vertices joined by an edge are said to be adjacent and each is incident with the edge. Welsh and Powell [46] point out the connection between the following CTT problem and the problem of coloring the vertices of a graph: Given

a set of teachers  $T = \{t_i\} \quad i=1, \dots, \alpha,$

a set of classes  $C = \{c_j\} \quad j=1, \dots, \beta,$

a set of hours  $H = \{h_k\} \quad k=1, \dots, \gamma,$

and an incompatibility matrix  $M$  with elements  $m_{jj'} = 0$  or  $1$  respectively, as classes  $c_j$  and  $c_{j'}$  can or cannot be assigned to the same hour. This includes such constraints as a teacher having to meet classes  $c_j$  and  $c_{j'}$ , in which case  $m_{jj'} = 1$ . Each class  $c_j$  refers to a course or lecture. The problem statement corresponds to real problems that often occur in universities.

In this association, classes correspond to vertices; hours correspond to colors; and, the condition that two classes which cannot be assigned to the same hour is represented by an edge joining the corresponding vertices in the graph. Before discussing graph colorations, the necessary terminology must be introduced.

Let  $G(V,E)$  denote a graph  $G$  that has vertex set  $V$  and edge set  $E$ . The degree  $d(v)$  of a vertex  $v$  is the number of edges incident with it. In a regular graph, all the vertices have the same degree. A complete graph  $K_m$  of order  $m$  has every pair of its vertices adjacent



and so is regular of degree  $m-1$ .

A walk in a graph is an alternating sequence of vertices and edges in  $G$ , beginning and ending with a vertex. Each edge is incident with the vertex preceding it and the vertex following it. A walk is often written  $v_1v_2\dots v_i$ ; the edges being evident by context. A path is a walk in which all vertices, and hence, edges are distinct. A closed walk has the same first and last vertices. A cycle is a closed walk  $v_1v_2\dots v_iv_1$ ,  $i \geq 3$ , in which the  $i$  vertices are distinct or a closed path.

In a connected graph, every pair of distinct vertices is joined by a path. A subgraph of  $G$  consists of subsets, of  $V$  and  $E$ , which themselves form a graph.

A graph is said to be  $n$ -colorable if each vertex can be assigned one of  $n$  or less colors in such a way that no two adjacent vertices have the same color. The chromatic number  $\chi(G)$  of a graph  $G$  is  $n$ , if  $G$  is  $n$ -colorable but not  $(n-1)$ -colorable.

An  $m \times m$  adjacency matrix  $A$  for a graph  $G(G,E)$  with  $|V| = m$  is defined as follows: The element in the  $(i,j)$  position of the matrix is 1 or 0 according to vertices  $v_i$  and  $v_j$  being joined by an



edge or not. The adjacency matrix is symmetric. Since graphs with loops are not being considered, the diagonal elements are all equal to 0. It will be assumed that distinct edges do not join the same pair of vertices. Hence, a graph can be defined by a set  $V$  of vertices and an adjacency matrix  $A$ .

The CTT problem defined is thus equivalent to determining an  $n$ -coloration of a graph  $G$  with a set of vertices  $V$ ,  $|V| = m$ , and adjacency matrix  $M = A$  where  $m = \beta$  and  $n = \gamma$ .

Assume all graphs are connected, since an arbitrary graph can always be decomposed into its connected components. Mathematically, this corresponds to finding the appropriate permutation matrix  $P$ . Matrix  $P$  has the property that  $P^{-1}AP$ , where  $A$  is the adjacency matrix, equals a matrix with block diagonal representation with diagonal submatrices containing all the nonzero elements of  $A$ .

The existence of an  $n$ -coloration of a graph  $G$  is related to determining the chromatic number  $\chi(G)$  of the graph  $G$ . As stated by Welsh and Powell, 'The problem of determining this chromatic number of a graph  $G$  is a well known unsolved problem'. This does require qualification since the case for  $n = 2$  was settled in the following theorem by König [27, pp. 8].

**Theorem 2.1** A graph  $G$  is 2-colorable if and only if no cycle in  $G$  has odd length.



Several bounds have been found of  $\chi(G)$  for an arbitrary graph  $G$ . Erdős<sup>11</sup> [27, pp. 31] points out that if  $K(G)$  denotes the number of vertices of the largest complete subgraph contained in  $G$ , then  $K(G) \leq \chi(G)$ . If  $G$  is an arbitrary graph with vertex set  $V$ , and  $D = \max_{v \in V} d(v)$ , the maximal degree, then  $G$  is  $(D+1)$ -colorable. This result was improved by Brooks [11] as given in the following theorem:

**Theorem 2.2** Let  $G$  be a connected graph, not a complete graph, and  $D$  its maximal degree. Then  $G$  is  $D$ -colorable.

Denote the degree of a vertex  $v_i$  of the graph  $G$  by  $d_i$ . Without loss of generality, assume that

$$d_1 \geq d_2 \geq \dots \geq d_m.$$

Welsh and Powell improved the upper bound of  $\chi(G)$  by showing that  $G$  is  $\alpha(G)$ -colorable for

$$\alpha(G) \geq \max_i \min(d_i + 1, i).$$

They also give an algorithm for finding such a  $\alpha(G)$ -coloration.

However, there are graphs for which  $\alpha(G) - \chi(G)$  may be arbitrarily large.

The previously discussed methods of Formby [20], Peck and Williams [41], and Williams [47] all apply to finding upper bounds of  $\chi(G)$ .



The above considerations pertain only to preventing vertices from being assigned the same color. Welsh and Powell state that:

'It does not answer the much more difficult problem, which occurs in practise, when in addition to an incompatibility matrix we are given a pre assignment matrix  $P = [p_{ij}]$  which specifies that certain jobs must be carried out on certain days ordained beforehand.'

It should be noted that a comparison of the CTT problem considered by Gotlieb and the CTT of Welsh and Powell gives rise to a hierarchy of CTT problems. Let the first of the problems be referred to as CTT(A) and the latter as CTT(B). The following shows that every CTT(A) problem can be formulated in terms of a CTT(B) problem, but that the converse is not true.

The following indicates how any CTT(A) problem may be written in terms of a CTT(B) problem. Let each meeting of every class in a CTT(A) problem correspond to a class in a CTT(B) problem. To illustrate, suppose each class  $c_j$ ,  $j=1, \dots, \beta$ , in a CTT(A) problem has  $\gamma$  meetings. Let  $c_{j_1}, \dots, c_{j_\gamma}$ , be the corresponding classes in the CTT(B) problem. From the definition of a CTT(A) problem, no two of these  $\gamma$  classes can meet at the same hour. Consider the incompatibility matrix  $M$  of the CTT(B) problem. All the elements of  $M$  that correspond to pairs of classes from the above set of  $\alpha$  classes must equal 1. Next, suppose that teacher  $t_i$  is to meet classes  $c_j$  and  $c_{j'}$ , in the CTT(A) problem, exactly



$r_{ij}$  and  $r_{ij'}$  times respectively. Without loss of generality, let  $c_{j1}, \dots, c_{jr_{ij}}$ , and  $c_{j'1}, \dots, c_{j'r_{ij'}}$ , be the corresponding classes, in the CTT(B) problem, which teacher  $t_i$  must meet. No two of these  $r_{ij} + r_{ij'}$  classes can meet at the same hour. Otherwise, teacher  $t_i$  must meet more than one class during some hour. Thus, all the elements of  $M$  that correspond to pairs of classes from this set of  $r_{ij} + r_{ij'}$  classes must equal 1. In general, this applies to any teacher and any pair of classes, in the CTT(A) problem, which he must meet. Thus, every CTT(A) problem may be written in terms of a CTT(B) problem.

The converse is not true: Examine a CTT(A) problem written in terms of a CTT(B) problem. Consider a pair of classes in the CTT(B) problem, that correspond to 'meetings' which belong to different classes in the CTT(A) problem. Furthermore, suppose that they are not met by the same teacher. Now set the requirement that this pair of classes is not to be assigned to the same hour. The problem cannot be stated in terms of CTT(A).

Thus, the CTT problems corresponding to CTT(B) problems are more general than those corresponding to CTT(A) problems. This previously unrecognized relationship between CTT(A) and CTT(B) problems will be referred to later.



### 2.1.5 Integer Programming Approaches

Hammer and Rudeanu [26], Harding (see [28]), and Zehnder [50] have formulated CTT problems as integer programs. However, as seen from the previous sections, the coloration of a graph is a fundamental subset of CTT problems. Consider the solution of the integer linear programs corresponding to the graph coloration problem.

Karp [30] showed that the general 0-1 integer programming problem and the determination of the chromatic number of a graph are equivalent, in the sense that either each of them possesses a polynomial-bounded algorithm, or neither of them does. At present no such algorithm is known. However, discussion will be given of two different, yet related, integer formulations. Graph coloring problems may be posed in either of these formulations.

First consider the hierarchy of covering problems (see [5]). Hall and Forman [24] suggested the use of a covering problem to find  $\chi(G)$ . The vertex coloration of a graph  $G(V,E)$  can be posed as a set covering problem. For example, let  $s_i$  be a subset of vertices in  $G$  such that all the vertices in  $s_i$  can be assigned to the same color. That is, no pair of vertices in  $s_i$  are joined by an edge. Such a subset  $s_i$  will be called a compatible subset. Let  $S$  be the set of subsets  $s_i$  in  $G$ . The problem is to find a



minimum family of subsets in  $S$  such that every vertex of  $G$  is contained in some subset of the family.

As an integer program, the problem is:

$$\min \sum_{j=1}^J x_j$$

subject to

$$\sum_{j=1}^J \alpha_j x_j = \alpha_0, \text{ and}$$

$$x_j = 0 \text{ or } 1, j = 1, \dots, J,$$

where  $\alpha_0$  is a column of  $J$  1's,

each  $\alpha_j = (a_{1j}, \dots, a_{mj})^T$  has  $a_{ij} = 1$  if  $s_i \in S$  and  $a_{ij} = 0$  otherwise,

$$j = |S|, \text{ and}$$

$$m = |V|.$$

The matrix  $A = (\alpha_1, \dots, \alpha_j)$ , with one column for each subset  $s_i \in S$  and one row for each vertex in  $V$ , is the  $(0,1)$  incidence matrix of vertices versus subsets. As compared to the simple covering problem where each  $\alpha_j$  has exactly two nonzero entries (equal to 1), the  $\alpha_j$  in the set covering problem contain an arbitrary number of 1's.



However, unlike the simple covering problem, the set covering problem has defied consistently efficient treatments (see [5]).

No generalization of the ideas used in direct and efficient algorithms for the simple covering have proven themselves for the set covering problems. Balinski [4] states that 'The discovery of a computationally efficient algorithm for solving the general covering problem would truly be a major contribution'.

There is another difficulty in posing the graph coloring problem as a set covering problem. Determining the set  $S$  of all subsets  $s_i$  is a major task. Approximations (see [24]) must very quickly be introduced in practice, due to the vast number of subsets  $s_i$ .

The second formulation of integer programs relates to the hierarchy of problems associated with networks. The formulation frequently stated in the literature [7,14,29] for the existence of face 4-colorations of planar graphs is the same for any coloration problem: Let each vertex  $v_i$  of  $G(V,E)$  with  $|V| = m$  be represented by a variable  $x_i$  where  $x_i$  can take on integer values  $0,1,\dots,n-1$ . For two adjacent vertices  $v_i$  and  $v_j$ ,  $x_i \neq x_j$ . The constraint may be stated as  $x_i - x_j \geq 1$  or  $x_j - x_i \geq 1$ . This pair of constraints do not have to be satisfied simultaneously. Defining a function  $f(x) = f(x_1, \dots, x_m)$  such that  $f(x) = 0$  if  $x$



represents a  $n$ -coloration and  $f(x) > 0$  otherwise, and rewriting the above pair of constraints as suggested by Hu [29] so as to obtain a convex solution space. Then the integer formulation is

$$\begin{array}{ll}
 \min f(x) & \\
 \text{subject to} & \\
 -x_i + x_j - n\delta_{ij} \leq -1 & \text{For all edges in } G, \text{ each} \\
 x_i - x_j + n\delta_{ij} \leq n - 1 & \text{coincident with a pair of} \\
 & \text{vertices } v_i \text{ and } v_j \text{ in } G. \\
 \delta_{ij} \leq 1 & \\
 x_i \leq n - 1 & i = 1, \dots, m. \\
 x_i \geq 0 & \\
 \delta_{ij} \geq 0 & \text{For each edge incident with} \\
 & \text{vertices } v_i \text{ and } v_j \text{ in } G.
 \end{array}$$

If the minimal value of  $f(x)$  becomes zero, then an  $n$ -coloration of  $G$  exists. If the minimal value of  $f(x)$  is greater than zero, then an  $n$ -coloration of  $G$  does not exist.

Consider the following definition.

**Definition 2.1:** A matrix  $A$  is said to be totally unimodular if, and only if, every subdeterminant of  $A$  equals  $+1$ , or  $0$ .



The following theorem was proved by Hoffman and Kruskal [29, pp. 125].

Theorem 2.3: A necessary and sufficient condition for the existence of an integer optimum solution of a linear program with constraints of the form  $Ax \leq b$  and  $x \geq 0$  is that the constraint matrix  $A$  is totally unimodular.

Corresponding to any edge in  $G$  and one of the vertices incident to that edge, consider the following two rows of the constraint matrix of the above formulation:

$$-x_i + x_j - n_{ij} \leq -1$$

$$x_i \leq -1.$$

These rows demonstrate a subdeterminant not equal to 0 or  $\pm 1$ . Namely,

$$\begin{vmatrix} -1 & -n \\ 1 & 0 \end{vmatrix} = n.$$

Of course to avoid triviality,  $n > 1$  is assumed. Hence, the above integer linear program does not reduce to a linear programming formulation that has an integer optimum solution.

It should be noted that there are network oriented integer



programs whose constraint matrices are not totally unimodular, but for which efficient algorithms exist. One example is the matching problem where a maximum matching is to be found. A matching in a graph is a subset of edges in  $G$  such that no two meet the same vertex in  $G$ . The constraint matrices are totally unimodular only if the graph  $G$  is bipartite; that is, the vertices of  $G$  can be partitioned into two parts so that each edge of  $G$  meets exactly one vertex in each part. Edmonds [18,19] and Witzhall and Zahn [49] found combinatorial algorithms for matching problems corresponding to non-bipartite graphs.

#### 2.1.6 Dynamic Programming Approach

Again, as in the previous section, consider  $G(V,E)$  with  $|V| = m$ .

Let

1.  $x_i$  be the variable relating vertex  $v_i$  of  $G$  to the color to which  $v_i$  is (to be) assigned,  $i = 1, \dots, m$ ;
2.  $X_i$  be the definition set of each  $x_i$ , defined to be the integers  $1, \dots, n$ ;
3.  $X$  be the Cartesian product of all  $X_i$ ;
4.  $E$  be a subset of the finite set  $X$  such that  $(x_1, \dots, x_m) \in E$  if, and only if,  $(x_1, \dots, x_m)$  represents an  $n$ -coloration of  $G$ ;



5.  $f(x_1, \dots, x_m)$  be a real function of  $m$  variables  $x_i$ ,  $i = 1, \dots, m$ ,  $f(x_1, \dots, x_m) = 0$  where  $(x_1, \dots, x_m)$  represents an  $n$ -coloration of  $G$ ; and,  $f(x_1, \dots, x_m) > 0$  otherwise.

Consider the following definitions given by Bonzon [9].

Definition 2.2: A discrete optimization problem is the search for a sequence  $(x_1^+, \dots, x_m^+)$ , called the optimal solution, giving the function  $f(x_1, \dots, x_m)$ , called the objective function, its global minimal value over the set of sequences  $(x_1, \dots, x_m) \in E$  called feasible solutions.

Definition 2.3: A graph of constraint is the subset  $E \subseteq X$ .

Definition 2.4: Sucessive projectives of the graph of constraint  $E$  are the sets  $P_i = \{(x_1, \dots, x_m) : \text{there exists } x_{i+1} \mid (x_1, \dots, x_i, x_{i+1}) \in P_{i+1}\}$ ,  $i = m-1, \dots, 1$ , with  $P_m = E$ .



Definition 2.5: Successive cuts of the graph of constraints

$E$  by a given sequence  $(x_1, \dots, x_{i-1})$  are the sets  $C_i(x_1, \dots, x_{i-1}) = \{x_i : \{(x_1, \dots, x_{i-1}, x_i) \in P_i\}\}$ ,  $i = m, \dots, 2$ , with  $C_1 = P_1$ .

Definition 2.6: A chained graph is a graph of constraint in

which the successive cuts by a given sequence

$(x_1, \dots, x_{i-1}) \in P_{i-1}$  do not depend effectively on all components of the sequence, but only on the last one, i.e.,  $C_i(x_1, \dots, x_{i-1}) \equiv E_i(x_{i-1})$ , for any  $(x_1, \dots, x_{i-1}) \in P_{i-1}$ .

The required result by Bonzon [9] can now be stated.

Theorem 2.4: A necessary condition for a discrete deterministic optimization problem to be solved by dynamic programming is that the graph of constraint be a chained graph.

In determining an  $n$ -coloration (or the existence thereof)

of a graph  $G$ , generally there does not exist a sequence

$(x_1, \dots, x_{i-1}) \in P_{i-1}$  such that the corresponding graph of constraint is a chained graph. That is, the color to which any vertex can be assigned cannot be assumed to depend on only one other vertex. For example, consider the following graph, and let  $E$  be the set of





Figure 2.1 Graph G

all  $n$ -colorations of  $G$ , regardless of how the vertices of  $G$  are labelled. Then:

$$P_4 = E;$$

$$P_3 = \{(x_1, x_2, x_3): \text{there exists } x_4 \mid x_4 \neq x_1, x_4 \neq x_2, x_4 \neq x_3\};$$

$$P_2 = \{(x_1, x_2): \text{there exists } x_3 \mid x_3 \neq x_1, x_3 \neq x_2\};$$

$$P_1 = \{(x_1): \text{there exists } x_2 \mid x_2 \neq x_1\};$$

$$C_4(x_1, x_2, x_3) = \{x_4: (x_1, x_2, x_3, x_4) \in P_4\} = E_4(x_1, x_2, x_3);$$

$$C_3(x_1, x_2) = \{x_3: (x_1, x_2, x_3) \in P_3\} = E_3(x_1, x_2);$$

$$C_2(x_1) = \{x_2: (x_1, x_2) \in P_2\} = E_2(x_1); \text{ and,}$$

$$C_1 = P_1.$$

Clearly, the graph of constraint is not a chained graph.

Thus, neither the question of existence nor the determination of an  $n$ -coloration of a graph can be answered in general by dynamic programming.

## 2.2 Conclusions

As for the existence of a solution to the CTT problem, it has been shown that a fundamental problem is the existence of an  $n$ -coloration for an arbitrary graph  $G(V, E)$  with  $|V| = m$ . If  $n \geq m$ ,  $n \geq B_U$  where  $B_U$  is an upper bound of  $\chi(G)$ , or if  $G$  is



known to be a complete graph, then the question of existence is answered. Otherwise, the question of existence remains a difficult problem. The only established upper bounds on the order of computation are those corresponding to enumeration and the algorithm of Turksen and Holzman. These bounds are in the order of  $n^m$ . If preassignment constraints are imposed upon a coloration problem, then the question of existence remains entirely unanswered.

In practice, heuristics are for the most part the only alternative for finding a solution to CTT problems. However, any deterministic tools which may be used with heuristics would be beneficial to practitioners. Graph theory and combinatorial analysis have provided the most powerful tools available. Of course, the need for more deterministic methods depends upon the problem. For example, it is intuitive that the closer  $n$  is to  $\chi(G)$ , the more difficult it is to find an  $n$ -coloration. The question is trivial when  $n \geq m$ .

The work in the subsequent chapters has been reported in terms of graphs. The main objectives of the research reported can be stated as follows:

1. To improve on the number of cases when the determination of the existence of an  $n$ -coloration of a graph can be realized.



2. To make an extension to the question of the existence of an  $n$ -coloration of a graph, so as to include graphs with pre-assignment constraints.

3. To make an extension to the question of the existence of an  $n$ -coloration of a graph, so as to include graphs with constraints that prevent certain vertices from being assigned to certain colors. Both the number of colors and the colors themselves, to which a given vertex is not to be assigned, may vary among the vertices. Again, these constraints correspond to requirements that occur in practical CTT problems.

4. To provide the basis for a more flexible means of determining an  $n$ -coloration of a graph, assuming that such a coloration exists.



## CHAPTER 3

### THE EXISTENCE OF COLORATIONS

In this chapter, the existence of an  $n$ -coloration of a graph is discussed. An upper bound of  $\chi(G)$  is given in Section 3.1. Also, the inherent weakness of any upper bound of  $\chi(G)$  is pointed out. A necessary condition for  $\chi(G)$  to be greater than  $K(G)$  is given in Section 3.2. Then in Section 3.3, a general method for determination of the existence is given; this method is directed at cases when  $B_L \leq n \leq B_U$ , where  $B_L$  and  $B_U$  are lower and upper bounds of  $\chi(G)$  respectively. These results are used to find properties of the adjacency matrix of a graph. These properties are sufficient for the number of steps required by the method to be bounded by a polynomial, or almost so. These properties are reported in Section 3.3. The most significant property identified is called the triangle property. The triangle property is related to the class of triangulated graphs in Section 3.4. Some ideas related to the triangle property are used to prove a result in graph theory. This result is given in Section 3.4.2.

#### 3.1 Upper Bounds of $\chi(G)$

Upper bounds  $B_U$  of  $\chi(G)$  can be useful for determining the existence of an  $n$ -coloration of a graph  $G$ . If  $n \geq B_U$ , then an



$n$ -coloration of  $G$  is known to exist. Known upper bounds for  $\chi(G)$ , such as those given by Brooks [11] and Welsh and Powell [46], are not necessarily the best possible for all cases. To illustrate, a procedure for finding another upper bound of  $\chi(G)$  is given below.

Consider the following: The partition number  $\pi = \pi(G)$  is the minimum number of vertex disjoint complete subgraphs of  $G$  that cover the vertices of  $G$ . The complement  $\bar{G}$  of a graph  $G$  has the same vertices as  $G$ . Two vertices are adjacent if, and only if, they are not adjacent in  $G$ . Let  $\pi$  be the partition number of  $\bar{G}$ . Nordhaus [38] proved the following theorem.

**Theorem 3.1** For any graph  $G$ ,  $\chi(G) = \pi$  and  $\chi(\bar{G}) = \pi$ .

Let  $\bar{S}$  be the set of all complete subgraphs in  $\bar{G}$ . Let  $\bar{S}_1$  be an element in  $\bar{S}$  such that the order of  $\bar{S}_1$  is greater than or equal to the order of any other element of  $\bar{S}$ . Clearly,  $\bar{S}_1$  is not a null graph. Similarly, define  $\bar{S}_2$  to be an element of  $\bar{S} - \{\bar{S}_1\}$  - {all the complete subgraphs in  $\bar{S}$  which are not vertex disjoint with  $\bar{S}_1$ }. Continuing in this manner, define a sequence  $\bar{S}_i$  of vertex disjoint complete subgraphs such that  $\bar{S}_i = \bar{G}$ . Since  $G$  has a finite vertex set, there exists a finite integer  $\pi'$  such that  $\bar{S}_i = \emptyset$  for  $i > \pi'$  and  $\bar{S}_i = \emptyset$  for  $i \leq \pi'$ .

Then  $\pi'$  is an upper bound of  $\chi(G)$ . The method also provides an



$n$ -coloration of  $G$  for  $n \geq \pi'$ . There do exist graphs for which  $\pi'$  is an improvement over the bounds of Brooks [11], and Welsh and Powell [46]. A significant point is that no upper bound can be considered to be consistently good. However to indicate an example, the graph in Cole's [11] example is a case where  $\pi'$  is an improvement over previous upper bounds. Welsh and Powell [46] show that their upper bound equals 14 for Cole's example. Welsh and Powell also show that Brook's upper bound equals 20. Applying the above procedure to Cole's example, the upper bound  $\pi'$  was found to equal 9. Thus, Cole's example may be documented as a case where  $\pi'$  is an improvement over previous upper bounds. If the number  $n$  of available colors is such that  $n < B_U$ , then no conclusion from the upper bound  $B_U$  as to the existence of an  $n$ -coloration can be made. This is true, regardless of how good any upper bound  $B_U$  of  $\chi(G)$  may be.

### 3.2 Lower Bounds of $\chi(G)$

As stated earlier, the best known lower bound of  $\chi(G)$  is  $K(G)$ , the order of the largest complete subgraph in  $G$ . Also, there exists graphs for which the difference  $\chi(G) - K(G)$  is arbitrarily large. A condition that must hold if  $\chi(G)$  is to be greater than  $K(G)$  is illustrated below.



Before stating Theorem 3.2, the following lemma is proved.

Lemma 3.1: Given  $G(V,E)$  and  $G$  not a complete graph. If  $\chi(G) > K(G) + k$ ,  $k = 0, 1, 2, \dots$ , then there exists a vertex  $v \in V$  such that degree  $d(v) = K(G) + k$ .

Proof: Suppose there does not exist  $v \in V$  such that  $d(v) \geq K(G) + k$  for some  $k = 0, 1, \dots$ . Then for any  $v \in V$ ,  $d(v) < K(G) + k$ . From Theorem 2.2,  $\chi(G) < K(G) + k$ . Thus, there is a contradiction. Q.E.D.

Corresponding to any  $v_i \in V$  of a graph  $G(V,E)$ , define  $V_i$  where

$$V_i = \{v \mid v \in V \text{ and } v \text{ adjacent to vertex } v_i\}.$$

Theorem 3.2: Given a graph  $G(V,E)$  which is not a complete graph.

Then there exists  $v_i \in V$  such that  $d(v_i) > K(G)$ .

Furthermore, there does not exist  $V_i' \subsetneq V_i$  such that there exists a complete subgraph  $G'(V_i', E')$  with  $|V_i'| = K(G)$ .

Proof: From Lemma 3.1, there exists  $v_i \in V$  such that  $d(v_i) \geq K(G)$ . Now suppose  $V_i' \subsetneq V_i$  such that  $|V_i'| = K(G)$  and there exists a complete subgraph  $G'(V_i', E')$  in  $G$ . Since  $v_i \notin V_i'$ , then  $v_i \notin V_i'$ . Also,  $v_i$  is adjacent to every vertex  $v \in V_i$ ; and hence,



$v \in V_1^1$ . Then there exists a complete subgraph of order  $K(G) + 1$  in  $G$ . However, this is a contradiction. Q.E.D.

The above condition is necessary for  $\chi(G)$  to be greater than  $K(G)$ . However, as no condition which is both necessary and sufficient has as yet been established, the best known lower bound of  $\chi(G)$  remains as  $K(G)$ . That is, if  $n \geq K(G)$ , then the existence of an  $n$ -coloration is indeterminate from  $K(G)$ .

### 3.3 Sufficiency Properties for Feasible Computation

Corresponding to CTT problems, one is typically given an adjacency matrix  $A$ , and the problem is to determine the existence of an  $n$ -coloration of the corresponding graph. Tools for the determination of the existence of an  $n$ -coloration are required in cases of indeterminacy. Such cases, as discussed in the previous two cases, arise when  $K(G) \leq n < B_u$  where  $B_u$  is the best known upper bound of  $\chi(G)$ .

A general method for establishing the existence of an  $n$ -coloration of a graph is given in Sections 3.3.1 and 3.3.2. A property of the adjacency matrix  $A$ , called the triangle property, is identified in Section 3.3.3. This property is sufficient for the number of steps required to be bounded by a polynomial. Cases are discussed, in Section 3.3.4, when the order of computation



required by the method presented is exponential. These cases are exponential to a much smaller degree compared to the degree for the most general case discussed in Section 3.3.2. An example, in Section 3.3.5, demonstrates the use of the results to determine the existence of an  $n$ -coloration. The order of computation required to determine the existence of an  $n$ -coloration using the results presented is compared to that using a multi-stage decision process.

### 3.3.1 Conditions Under Which An $n$ -coloration Exists

Given a graph  $G(V,E)$  with  $|V| = m$ . If there are  $n$  available colors and if all the edges of  $G$  are neglected, then there are  $n^m$  possible colorations of  $G$ . Each time a non-adjacent pair of vertices in  $G$  is adjoined by an edge, there is a possible reduction in the number of  $n$ -colorations of  $G$ . In order to facilitate counting the total number of  $n$ -colorations eliminated by the edges in  $E$ , the  $n$ -colorations of  $G$  can be placed in one to one correspondence with a set of paths in a tree. The following discussion formalizes such a correspondence.

Before proceeding, some terminology is required. A graph with no cycles is acyclic. A tree is a connected acyclic graph. A rooted tree is a tree with one of its vertices distinguished from the others by being called the root. Let  $R$  refer to the root vertex. A terminal



vertex of a rooted tree is any vertex  $v$  such that the degree of  $v$  equals 1. A complete path in a rooted tree  $T$  is a path such that its first vertex coincides with the root vertex  $R$ . The number of edges in a path is called its length.

Construct a rooted tree  $T$  whose vertices are labelled, though not uniquely, in the following recursive manner. Let  $T$  have a root vertex  $R$ . Let there be  $n$  vertices, labelled  $x_{11}, \dots, x_{1n}$ , adjacent to vertex  $R$ . Then let there be  $n$  vertices, labelled  $x_{i1}, \dots, x_{in}$ , adjacent to each of the vertices labelled  $x_{i-11}, \dots, x_{i-1n}$  for  $i = 2, \dots, m$ . Note that the vertices of  $T$  are not uniquely labelled. Define  $P'$  as the set of all complete paths of length  $m$  in  $T$ . For complete paths  $p$  and  $p'$  in  $T$ , where  $p = (R, x_{1j_1}, \dots, x_{mj_m})$  and  $p' = (R, x_{1j'_1}, \dots, x_{mj'_m})$ , define  $p \neq p'$  if and only if there exists at least one  $i$ ,  $1 \leq i \leq m$ ,  $j_i \neq j'_i$ . Note that  $|P'| = n^m$ .

Let  $G'$  be a graph with a set of vertices  $V$ , as in  $G$ , and a set of edges  $E = \emptyset$ . Label the  $m$  vertices of  $G$  and  $G'$  as  $v_1, \dots, v_m$ . Denote an  $n$ -coloration of the vertices  $v_1, \dots, v_m$  in  $G$  and  $G'$  as  $(x_{1j_1}, \dots, x_{mj_m})$  where  $x_{ij_i}$  denotes vertex  $v_i$  having been assigned to color  $c_k = c_{j_i}$ , and where  $\{c_k\}_{k=1}^n$  is the set of  $n$  colors available. Two colorations  $c = (x_{1j_1}, \dots, x_{mj_m})$  and  $c' = (x_{1j'_1}, \dots, x_{mj'_m})$  will be considered equal if and only if  $j_i = j'_i$  for  $i = 1, \dots, m$ .



Let sets  $C$  and  $C'$  correspond to all the  $n$ -colorations of the labelled graphs  $G$  and  $G'$ , respectively. Then  $|C'| = n^m$  and  $0 \leq |C| \leq n^m$ .  $G$  is  $n$ -colorable if and only if  $|C| > 0$ .

If  $c = (x_{1j_1}, \dots, x_{mj_m}) \in C'$ , a mapping  $\rho$  from  $C'$  to  $P'$  can be defined such that  $\rho(c) = (R, x_{1j_1}, \dots, x_{mj_m})$ , a complete path of length  $m$  in tree  $T$ . Let  $p = (R, x_{1j_1}, \dots, x_{mj_m})$  be any element in  $P'$ . Then there exists a coloration  $c \in C'$  such that  $\rho(c) = p$ . Thus,  $\rho$  is an onto mapping. Let  $c$  and  $c' \in C'$  such that  $c \neq c'$ ; then,  $\rho(c) \neq \rho(c')$ . Therefore,  $\rho$  is a one to one mapping. Hence, there is a one to one correspondence between the  $n$ -colorations of  $G$ ; and the complete paths of length  $m$  in  $T$ .

Because any coloration of  $G$  is a coloration of  $G'$ , then  $C \subseteq C'$ . Let  $\rho(C) = P$ . Then  $P \subseteq P'$  and  $|C' - C| = |P' - P|$ . Thus,  $G$  is  $n$ -colorable if and only if  $|C' - C| < n^m$ , or equivalently, if  $|P' - P| < n^m$ .

Thus, a correspondence between the  $n$ -colorations of  $G$  and a subset of all complete paths of length  $m$  in a tree has been established. As well, the condition under which an  $n$ -coloration of  $G$  exists has been given in terms of a subset of all complete paths in the tree; namely,  $|P' - P| < n^m$ .



### 3.3.2 Enumerative Properties of the Set $P' - P$

In Section 3.3.1, the existence of an  $n$ -coloration of  $G$  was stated in terms of the order of the subset  $P' - P$ .  $P' - P$  is a subset of the set of complete paths of length  $m$  in a tree  $T$  with  $n$  terminal nodes. The following discussion concerns the computation of  $|P' - P|$ .

Let  $A = [a_{ij}]$  be the  $m \times m$  adjacency matrix of a graph  $G(V, E)$ .

Let  $C_{ij} = \{c \mid c \in C; \text{ and } c \text{ such that vertices } v_i \text{ and } v_j \text{ of } G \text{ are assigned identical colors}\},$

and  $I = \{(i, j) \mid i > j \text{ and } a_{ij} = 1 \text{ in } A\}.$

Then,  $|I| = |E|.$

$C_{ij}$  is the set of colorations in  $C'$  that violate the condition corresponding to  $a_{ij} = 1$  in matrix  $A$ . Let  $\rho(C_{ij}) = P_{ij}$ .  $P_{ij}$ ,  $C_{ij}$ , and  $a_{ij} = 1$  will be referred to interchangeably in the above context.

Then  $|C' - C| = |\cup_{(i,j) \in I} C_{ij}| = |\cup_{(i,j) \in I} P_{ij}|.$

To illustrate the above, let  $G = G(V, E)$  with  $V = \{v_1, v_2, v_3\}$  and  $n = 2$ . Then the partially labelled tree  $T_1$  in Figure 3.1 with root node  $R$  and  $2^3 = 8$  terminal nodes corresponds to the tree  $T$ . The 8 complete paths of length 3 correspond to the 2-colorations of the graph  $G'(V, E')$  with  $E' = \emptyset$ . Now, consider  $C_{ij}$  with



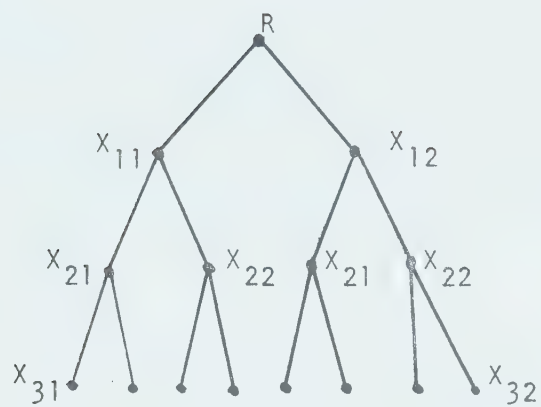


Figure 3.1 Tree  $T_1$

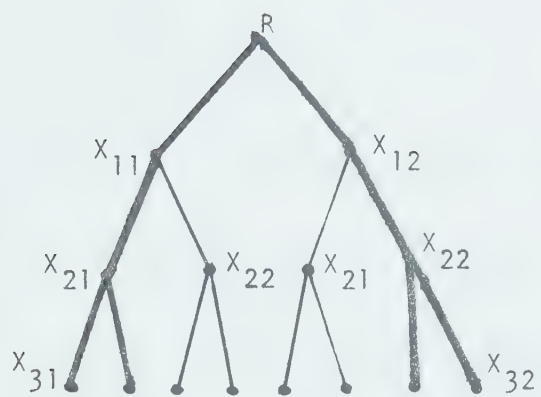


Figure 3.2 Tree  $T_2$



$i = 2$  and  $j = 1$ . Then  $C_{21}$  is the set of 2-colorations of  $G'$  such that vertices  $v_1$  and  $v_2$  are assigned identical colors. Figure 3.2 shows the complete paths of length 3 that correspond to those complete paths in the set  $\rho(C_{21}) = P_{21}$ .

Thus if  $|\bigcup_{(i,j) \in I} P_{ij}| < n^m$  then an  $n$ -coloration exists. Otherwise  $|\bigcup_{(i,j) \in I} P_{ij}| = n^m$  and no  $n$ -coloration exists. Next recall the following elementary set relations.

From set theoretic considerations, given sets  $S_1$ ,  $S_2$ , and  $S_3$  such that  $S_1 \cap S_2 = \emptyset$ , then  $|S_1 \cup S_2| = |S_1| + |S_2|$ . Define  $S_1 + S_2 = S_1 \cup S_2$  and  $S_1 - S_2 = \{s | s \in S_1, s \notin S_2\}$ . Then  $(S_1 - S_2) \cap S_3 = S_1 \cap S_3 - (S_2 \cap S_3)$ . Also if  $S_2 \subseteq S_1$ , then  $|S_1 - S_2| = |S_1| - |S_2|$ .

Denote the elements  $(i,j) \in I$  as  $\{(i_1, j_1), \dots, (i_{||I||}, j_{||I||})\}$ . Then from set theoretic considerations,

$$\begin{aligned} \bigcup_{(i,j) \in I} P_{ij} &= \{P_{i_1 j_1}\} \cup \{P_{i_2 j_2} - (P_{i_1 j_1} \cap P_{i_2 j_2})\} \\ &\quad \cup \{P_{i_3 j_3} - (P_{i_1 j_1} \cap P_{i_3 j_3}) - ((P_{i_2 j_2} \cap P_{i_3 j_3}) - \\ &\quad (P_{i_1 j_1} \cap P_{i_2 j_2} \cap P_{i_3 j_3}))\} \dots \\ &\quad \cup \{ \dots ((P_{i_1 j_1} \cap \dots \cap P_{i_{||I||} j_{||I||}}) \dots) \}. \end{aligned}$$

Thus

$$\begin{aligned} |\bigcup_{(i,j) \in I} P_{ij}| &= \{|P_{i_1 j_1}|\} + \{|P_{i_2 j_2} - |P_{i_1 j_1} \cap P_{i_2 j_2}|\} + \\ &\quad \{|P_{i_3 j_3} - |P_{i_1 j_1} \cap P_{i_3 j_3}| - (|P_{i_2 j_2} \cap P_{i_3 j_3}| - \\ &\quad |P_{i_1 j_1} \cap P_{i_2 j_2} \cap P_{i_3 j_3}|)\} \dots \\ &\quad \{ \dots ((|P_{i_1 j_1} \cap \dots \cap P_{i_{||I||} j_{||I||}}|) \dots) \}. \end{aligned}$$



To evaluate the above expression, the determination of  $|\cap_{(i,j) \in I'} P_{ij}|$ , where  $I' \subseteq I$ , requires consideration. Note that  $|P_{i_1 j_1} \cap P_{i_2 j_2}|$  is written as  $|\cap_{(i,j) \in I'} P_{ij}|$  with  $I' = \{(i_1, j_1), (i_2, j_2)\}$ .

Prior to giving several results concerning the determination of  $|\cap_{(i,j) \in I'} P_{ij}|$ , some preliminaries to facilitate proving those results are discussed. A labelling  $L$  of the vertices of the graph  $G$  will refer to a particular manner in which the vertices have been labelled. The sets  $\{P_{ij}\}_{(i,j) \in I'}$ ,  $I' \subseteq I$ , can be considered to correspond to specific sets of vertices  $V' \subseteq V$  and edges  $E' \subseteq E$  whereas the  $(i,j)$  correspond to a particular labelling  $L$  of the vertices in  $V$ .

Let  $I_{L_1}$  and  $I_{L_2}$  correspond to labellings  $L_1$  and  $L_2$  of  $V$  respectively. Let  $I'_{L_1} \subseteq I_{L_1}$  and  $I'_{L_2} \subseteq I_{L_2}$ .  $I'_{L_1}$  and  $I'_{L_2}$  will be said to be isomorphic and written  $I'_{L_1} \sim I'_{L_2}$  if there exists a one to one correspondence between the edges corresponding to elements in  $I'_{L_1}$  and those in  $I'_{L_2}$ . If  $I'_{L_1} \sim I'_{L_2}$  then

$$|\cap_{(i,j) \in I'_{L_1}} P_{ij}| = |\cap_{(i,j) \in I'_{L_2}} P_{ij}|.$$

That is, the order of the intersection of the set of complete paths that are of length  $m$  in  $T$ , that correspond to colorations, and that have been eliminated by a given set of edges, is independent of how the vertices have been labelled.

To illustrate, suppose  $n = 2$  and consider a graph  $G$  with  $|V| = 4$  with two labellings  $L_1$  and  $L_2$ . Suppose  $I'_{L_1} = \{(2,1), (3,2)\}$  and  $I'_{L_2} = \{(3,2), (4,3)\}$  correspond to the same two edges of  $G$ . The complete paths



of length 4 that correspond to the intersection of the elements  $I'_{L_1}$  and  $I'_{L_2}$  are indicated by Tree  $T_3$  in Figure 3.3 and by Tree  $T_4$  in Figure 3.4 respectively. Clearly

$$|\cap_{(i,j) \in I'_{L_1}} P_{ij}| = |\cap_{(i,j) \in I'_{L_2}} P_{ij}| = 4.$$

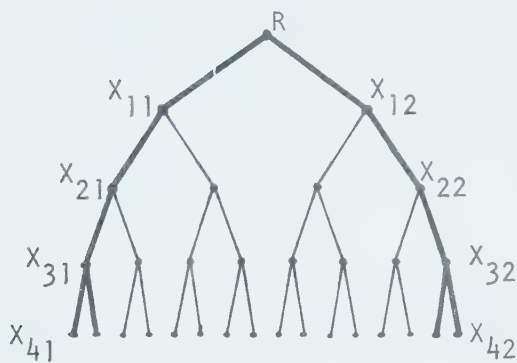


Figure 3.3 Tree  $T_3$

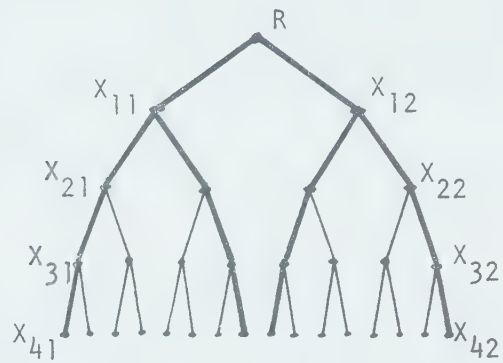


Figure 3.4 Tree  $T_4$

Define the cardinal set  $D_{I'_{L_1}}$  for  $I'_{L_1} \subseteq I_{L_1}$  as  $D_{I'_{L_1}} = \{d \mid \text{there exists } (i,j) \in I'_{L_1} \text{ such that } d = i \text{ or } d = j\}$ . Next relabel  $V$  with a labelling  $L_2$  such that  $I'_{L_2} \sim I'_{L_1}$  and  $D_{I'_{L_2}} = \{1, 2, \dots, |D_{I'_{L_1}}|\}$ . Such a relabelling is always possible: Arrange the  $|D_{I'_{L_1}}|$  elements of  $D_{I'_{L_1}}$  in ascending order, place them in one to one correspondence with the integers  $1, \dots, |D_{I'_{L_1}}|$ , and label the vertices accordingly.

Observe that  $|D_{I'_{L_1}}|$  equals the number of distinct rows (columns) of the adjacency matrix  $A$  such that each row (column) contains at least one element  $a_{ij} = 1$ ,  $i > j$  or  $i < j$ , that corresponds to some element  $(i,j) \in I'$ .



A labelling  $L_2$  has been defined. The cardinal set for  $I'_{L_2}$  has a particular property. Due to the following lemma, a further relationship between the two labellings  $L_1$  and  $L_2$  is shown to hold.

**Lemma 3.2** Given labellings  $L_1$  and  $L_2$  of  $V$  such that  $I'_{L_1} \sim I'_{L_2}$ , and

$$D_{I'_{L_2}} = \{1, 2, \dots, |D_{I'_{L_1}}|\}.$$

Then for each  $d$ ,  $1 \leq d \leq |D_{I'_{L_1}}|$ ,

there exists  $(i, j) \in I'_{L_2}$  such that  $d = i$ .

**Proof:** Define a permutation operation as follows:

A permutation operation on the adjacency matrix  $A$  of a graph  $G$  from position  $(p_1, p_2)$ ,  $p_1 > p_2$ , to position  $(p_3, p_4)$ ,  $p_3 > p_4$ , in  $A$  consists of

1. interchanging rows and columns  $p_2$  and  $p_4$ ,
2. interchanging rows and columns  $p_1$  and  $p_3$ , and
3. relabelling the vertices of  $G$  such that the subscripts of the elements of  $A$  correspond to row (column) numbers of  $A$ .

The effect of a permutation operation is a relabelling of the vertices of  $G$  so that the element  $a_{p_3 p_4}$  in  $A$  is 'moved' to the  $(p_1, p_2)$  position.

Suppose now that labelling  $L_2$  of  $V$  is such that there exists  $d$ ,  $1 \leq d \leq |D_{I'_{L_1}}|$ , such that there does not exist  $(i, j) \in I'_{L_2}$  for which  $d = i$ . Beginning at row 2 of the adjacency matrix  $A$ , perform as necessary a series of permutation operations on the first  $|D_{I'_{L_1}}|$  rows (columns) of  $A$  so as to get an element corresponding to at least one



$(i,j) \in I'_{L_2}$  in each of the rows  $2, \dots, |D_{I'_1}|$ . If for some row  $d$ ,  $1 \leq d \leq |D_{I'_1}|$ , there does not exist an element  $a_{p_1 p_2} = 1$ ,  $p_1 > p_2$ , corresponding to some  $(i,j) \in I'_{L_2}$  and no such element can be moved there by means of a permutation operation, then there does not exist an  $(i,j) \in I'_{L_2}$  such that  $d = i$  or  $d = j$ . But this is a contradiction. Hence the assumption is valid. Q.E.D.

Having given some preliminaries, a series of results related to the computation of  $|\bigcap_{(i,j) \in I'} P_{ij}|$  are stated and proved.

Lemma 3.3  $|P_{ij}| = n^{m-1}$ .

Proof: Without loss of generality, assume  $i = 2$  and  $j = 1$ . This can always be obtained by a series of permutation operations on the adjacency matrix  $A$ . By definition,

$$P_{ij} = \{p | p \in P' \text{ and } p \text{ passes through both vertices } x_{ik} \text{ and } x_{jk} \text{ for some integer } k, 1 \leq k \leq n\}.$$

From the definitions of the tree  $T$ , there are  $n^{m-2}$  elements of  $P_{ij}$  passing through vertices  $x_{1k}$  and  $x_{2k}$  for  $k = 1, \dots, n$ . Hence  $|P_{ij}| = n \cdot n^{m-2} = n^{m-1}$ . Q.E.D.

Defining  $m$ ,  $n$ , and labelling  $L_1, L_2$  with  $I'_{L_1}$  and  $I'_{L_2}$  as above, then Lemma 3.4 is a generalization of Lemma 3.3.

Lemma 3.4 Given  $\{P_{ij}\}_{(i,j) \in I'_{L_1}}$ , then  $|\bigcap_{(i,j) \in I'_{L_1}} P_{ij}| \geq n^{m+1 - |D_{I'_{L_1}}|}$ .



Proof: From previous discussion,  $|\cap_{(i,j) \in I'_{L_1}} P_{ij}| = |\cap_{(i,j) \in I'_{L_2}} P_{ij}|$ . Then given some  $k$ ,  $1 \leq k \leq n$ , there exists  $n^{m-|D_{I'_{L_1}}|}$  complete paths  $p$  in  $T$  that pass through vertices  $\{x_{i_k}\}_{i=1}^{|D_{I'_{L_1}}|}$  in  $T$ . Clearly  $p \in \cap_{(i,j) \in I'_{L_1}} P_{ij}$ . Since this holds for each  $k$ ,  $1 \leq k \leq n$ , then

$$|\cap_{(i,j) \in I'_{L_1}} P_{ij}| \geq n \cdot n^{m-|D_{I'_{L_1}}|} = n^{m+1-|D_{I'_{L_1}}|}. \quad \text{Q.E.D.}$$

In order to prove equality in the statement of Lemma 3.4, the following lemma is required.

Lemma 3.5 Given  $\{P_{ij}\}_{(i,j) \in I'_{L_1}}$  and  $P_{i_0 j_0}$ ,  $i_0 > j_0$ , such that either  $i_0$  or  $j_0 \notin D_{I'_{L_1}}$  or both  $i_0, j_0 \notin D_{I'_{L_1}}$ . Then

$$|(\cap_{(i,j) \in I'_{L_1}} P_{ij}) \cap (P_{i_0 j_0})| = (|\cap_{(i,j) \in I'_{L_1}} P_{ij}|)/n.$$

Proof: Again from previous discussion, there exists a labelling  $L_2$  of the vertices of the graph  $G$  such that  $I'_{L_2} \sim I'_{L_1}$ ,  $D_{I'_{L_2}} = \{1, 2, \dots, |D_{I'_{L_1}}|\}$ ,  $I'_{L_2} = \{(i_0, j_0)\} \sim I'_{L_2} = \{(i'_0, j'_0)\}$  and  $i'_0 = |D_{I'_{L_1}}| + 1$ . Then for each  $p \in \cap_{(i,j) \in I'_{L_2}} P_{ij}$  there exists a set of

vertices  $X_q = \{x_{dd_q}\}_{d \in D_{I'_{L_2}}}$  in  $T$  such that  $p$  passes through each of

the vertices in  $X_q$ . Consider two sets  $X_q$  and  $X_{q'}$  to be distinct if  $q \neq q'$  and there exists at least one  $d \in D_{I'_{L_2}}$  such that for the

corresponding subscripts  $d_q$  and  $d_{q'}$ ,  $d_q \neq d_{q'}$ . Then there exists a

set  $X$ , with index set  $Q$ , of distinct elements  $X = \{X_q\}_{q \in Q}$  such that



for any  $p \in \bigcap_{(i,j) \in I'_{L_2}} P_{ij}$ , there exists  $X_q \in X$  such that  $p$  passes through all of the vertices in  $X_q$ . Let  $P_{X_q}$  be the elements in  $\bigcap_{(i,j) \in I'_{L_2}} P_{ij}$  such that  $p \in P_{X_q}$  if and only if  $p$  passes through the vertices in  $X_q \in X$ . From the definitions of tree  $T$  and  $X_q$ ,

$|P_{i_0'j_0'} \cap P_{X_q}| = (|P_{X_q}|)/n$ . But since  $P_{X_q} \cap P_{X_{q'}} = \emptyset$  for  $q \neq q'$  and

$U_{q \in Q} P_{X_q} = \bigcap_{(i,j) \in I'_{L_2}} P_{ij}$ , then

$$\begin{aligned}
 |(\bigcap_{(i,j) \in I'_{L_2}} P_{ij}) \cap P_{i_0'j_0'}| &= |(U_{q \in Q} P_{X_q}) \cap P_{i_0'j_0'}| \\
 &= |U_{q \in Q} (P_{X_q} \cap P_{i_0'j_0'})| \\
 &= \sum_{q \in Q} |P_{X_q} \cap P_{i_0'j_0'}| \\
 &= \sum_{q \in Q} (|P_{X_q}|)/n \\
 &= (|U_{q \in Q} P_{X_q}|)/n \\
 &= (|\bigcap_{(i,j) \in I'_{L_2}} P_{ij}|)/n. \quad \text{Q.E.D.}
 \end{aligned}$$

With lemma 3.5, then the following desired results are proved.

**Theorem 3.3** Given  $\{P_{ij}\}_{(i,j) \in I'_{L_1}}$ , then  $|\bigcap_{(i,j) \in I'_{L_1}} P_{ij}| = n^{m+1-|D_{I'_{L_1}}|}$ .

**Proof:** There exists a labelling  $L_2$  such that  $I'_{L_2} \sim I'_{L_1}$  and

$D_{I'_{L_2}} = \{1, 2, \dots, |D_{I'_{L_2}}|\}$ . Unless  $I'_{L_1} = \emptyset$ , then  $|D_{I'_{L_1}}| \geq 2$ . If  $|D_{I'_{L_1}}| = 2$



then  $D_{I'_{L_2}} = \{(1,2)\}$  and from Lemma 3.3,  $|P_{21}| = n^{m-1} = n^{m+1-|D_{I'_{L_1}}|}$ .

By induction on  $|D_{I'_{L_1}}| > 2$  and repeated invocation of Lemma 3.5

the desired result is obtained. Q.E.D.

Corollary 3.1 Given  $\{P_{ij}\}_{(i,j) \in I'_{L_1}}$  and  $P_{i_0 j_0}$  such that  $i_0, j_0 \in D_{I'_{L_1}}$ ,

then

$$|\bigcap_{(i,j) \in I'_{L_1}} P_{ij} \cap P_{i_0 j_0}| = |\bigcap_{(i,j) \in I'_{L_1}} P_{ij}|.$$

Proof: Let  $I''_{L_1} = I'_{L_1} \cup (i_0, j_0)$ . Since  $i_0, j_0 \in D_{I'_{L_1}}$ , then

$|D_{I'_{L_1}}| = |D_{I''_{L_1}}|$ . Hence the result follows from Theorem 3.3. Q.E.D.

Thus  $|\bigcap_{(i,j) \in I'} P_{ij}|$  can be evaluated for any  $I' \subseteq I$  and hence using the expression of  $|U_{(i,j) \in I} P_{ij}|$ , given early within this section, then  $|U_{(i,j) \in I} P_{ij}|$  can be evaluated in order to determine the existence of an  $n$ -coloration.

Consider the order of computation required to evaluate

$|U_{(i,j) \in I} P_{ij}|$ . The expression for  $|U_{(i,j) \in I} P_{ij}|$  contains  $2^{|I|} = 2^{|E|}$  terms of the form  $|\bigcap_{(i,j) \in I'} P_{ij}|$ ,  $I' \subseteq I$ .  $|I'|$  is the order of computation required to evaluate a term  $|\bigcap_{(i,j) \in I'} P_{ij}|$ . Since  $|I'| \leq |E|$ , the order of computation to evaluate all the terms is  $|E| \cdot 2^{|E|} \approx 2^{|E|}$ . The number of additions (subtractions) of terms  $|\bigcap_{(i,j) \in I'} P_{ij}|$  is in the order of  $2^{|E|}$ . Hence, the total order of computation required to evaluate  $|U_{(i,j) \in I} P_{ij}|$  is  $2^{|E|}$ .



Though the order of computation for determination of the existence of an  $n$ -coloration using the above results is exponential, there exist many graphs for which  $2^{|E|} \ll n^m$ , particularly if the adjacency matrix is sparse in non-zero elements and/or if  $n$  is large.

### 3.3.3 More Enumerative Properties of $P' - P$

So as to display cases when the order of computation of  $|U_{(i,j) \in I} P_{ij}|$  is algebraic, further consideration is given to enumerative properties of  $P' - P$  in this section.

Given  $I$  as previously defined, define  $I_{i_0}$  where

$$I_{i_0} = \{(i,j) \mid \text{some } (i,j) \in I \text{ with } i = i_0\}.$$

Next suppose  $|U_{(i,j) \in I'} P_{ij}|$  has been determined for  $I' \subseteq I$  and

$|U_{(i,j) \in I'} P_{ij} \cup U_{(i,j) \in I_{i_0}} P_{ij}|$  where  $I_{i_0} \subseteq I$  and  $i_0 \notin D_{I'}$  is to be determined. Denote the elements of  $I_{i_0}$  as  $\{(i_0, j_1), \dots, (i_0, j_{|I_{i_0}|})\}$ .

Then from set theoretic considerations,

$$\begin{aligned} & \{U_{(i,j) \in I'} P_{ij} \cup U_{(i,j) \in I_{i_0}} P_{ij}\} = \\ & \{U_{(i,j) \in I'} P_{ij} \cup \{P_{i_0 j_1} - (P_{i_0 j_1} \cap U_{(i,j) \in I'} P_{ij})\} \\ & \quad \vdots \\ & \quad \{P_{i_0 j_1} \cap \dots \cap P_{i_0 j_{|I_{i_0}|}} \cap U_{(i,j) \in I'} P_{ij} \dots\}\}. \end{aligned}$$



Thus

$$\begin{aligned}
 & |(U_{(i,j) \in I', P_{ij}}) \cup (U_{(i,j) \in I_0, P_{ij}})| = \\
 & \{|U_{(i,j) \in I', P_{ij}}| + \{|P_{i_0 j_1}| - |P_{i_0 j_1} \cap (U_{(i,j) \in I', P_{ij}})|\} + \\
 & \quad \vdots \\
 & \quad |P_{i_0 j_1} \cap \dots \cap P_{i_0 j_{|I_0|}}| \cap (U_{(i,j) \in I', P_{ij}})| \dots\}.
 \end{aligned}$$

For any  $(i_0, j) \in I_0$ ,  $|(U_{(i,j) \in I', P_{ij}}) \cup (P_{i_0 j})| =$

$$|U_{(i,j) \in I', P_{ij}}| + |P_{i_0 j}| - |P_{i_0 j} \cap (U_{(i,j) \in I', P_{ij}})|.$$

Next the following theorem is proved.

**Theorem 3.4** Given  $\{P_{ij}\}_{(i,j) \in I'}$ ,  $I' \subseteq I$ , and  $P_{i_0 j_0}$ ,  $(i_0, j_0) \in I$  and

$$\begin{aligned}
 & i_0 \notin D_{I'}, \text{ then } |(U_{(i,j) \in I', P_{ij}}) \cup P_{i_0 j_0}| = \\
 & |U_{(i,j) \in I', P_{ij}}| + (n^m - |U_{(i,j) \in I', P_{ij}}|)/n.
 \end{aligned}$$

**Proof:** Let  $I'' = \{(i_0, j_0)\}$ . From previous discussion regarding the labelling of vertices, it can be assumed that  $D_{I'} = \{1, \dots, |D_{I'}|\}$ . Also, it can be assumed  $D_{I''} = \{k, |D_{I'}| + 1\}$  for some  $k$ ,  $1 \leq k \leq |D_{I'}|$  and  $i_0 = |D_{I'}| + 1$ .

For each  $p \in P' - U_{(i,j) \in I', P_{ij}}$ , let

$$\begin{aligned}
 X_p &= \{x_{ss_p} \mid x_{ss_p} \in T \text{ for every } s \in D_{I'}, \text{ and } s_p \text{ such that} \\
 & p \text{ passes through } x_{ss_p}\}.
 \end{aligned}$$

Furthermore, let  $X = \{X_r\}_{r \in R}$  be a set of distinct representatives for



the elements of  $P' - \bigcup_{(i,j) \in I} P_{ij}$  with suitable index set  $R$  such that

1. for every  $p \in P' - \bigcup_{(i,j) \in I} P_{ij}$  there exists a  $X_r$ ,  $r \in R$ , such that  $p$  passes through the vertices in  $X_r$ , and

2. if  $r \neq r'$ , then there exists  $x_{ss_r} \in X_r$  and  $x_{ss_{r'}} \in X_{r'}$  such that  $s_r \neq s_{r'}$ .

Define  $P_{X_r} = \{p \mid p \in P' - \bigcup_{(i,j) \in I} P_{ij} \text{ such that } p \text{ passes through the vertices in } X_r\}$

for every  $r \in R$ . From the definition of  $P_{X_r}$  and the tree  $T$ ,

$|P_{X_r}| = |P_{X_{r'}}|$  and  $P_{X_r} \cap P_{X_{r'}} = \emptyset$  for  $r, r' \in R$ ,  $r \neq r'$ . Also

$\bigcup_{r \in R} P_{X_r} = P' - \bigcup_{(i,j) \in I} P_{ij}$ . For each  $P_{X_r}$ ,  $r \in R$ , define

$P_{X_r+x_{i_0k}} = \{p \mid \text{all } p \in P_{X_r} \text{ such that } p \text{ passes through vertex } x_{i_0k}\}$

for  $k = 1, \dots, n$ . Then  $P_{X_r} = \bigcup_{k=1}^n P_{X_r+x_{i_0k}}$ . For  $k, k'$ ,  $k \neq k'$

and  $1 \leq k, k' \leq n$ ,  $|P_{X_r+x_{i_0k}}| = |P_{X_r+x_{i_0k'}}|$  and

$P_{X_r+x_{i_0k}} \cap P_{X_r+x_{i_0k'}} = \emptyset$ . There exists  $k$ ,  $1 \leq k \leq n$ ,

$$|P_{X_r} \cap P_{i_0j_0}| = |P_{X_r+x_{i_0k}}| = |P_{X_r}|/n.$$

These equations are immediate from the definition of tree  $T$ ,  $i_0 =$

$|D_{i_1}| + 1$ , and  $D_{i_1} = \{1, \dots, |D_{i_1}|\}$ . Thus

$$\begin{aligned} |P_{i_0j_0} \cap (P' - \bigcup_{(i,j) \in I} P_{ij})| &= |P' - \bigcup_{(i,j) \in I} P_{ij}|/n \\ &= (|P'| - |\bigcup_{(i,j) \in I} P_{ij}|)/n. \end{aligned}$$



Therefore  $|(U_{(i,j) \in I', P_{ij}}) \cup P_{i_0 j_0}| = |U_{(i,j) \in I', P_{ij}}| + (n^m - |U_{(i,j) \in I', P_{ij}}|)/n$ . Q.E.D.

The following corollary is immediate from Theorem 3.4.

Corollary 3.2 Given  $\{P_{ij}\}_{(i,j) \in I'}$ ,  $I' \subseteq I$ , and  $P_{i_0 j_0}$ ,  $(i_0, j_0) \in I$  and  $i_0 \notin D_{I'}$ , then

$$|(U_{(i,j) \in I', P_{ij}}) \cap P_{i_0 j_0}| = |P_{i_0 j_0}| - (n^m - |U_{(i,j) \in I', P_{ij}}|)/n.$$

$$\text{Proof: } |P_{i_0 j_0}| - |P_{i_0 j_0} \cap (U_{(i,j) \in I', P_{ij}})| =$$

$$|(U_{(i,j) \in I', P_{ij}}) \cup P_{i_0 j_0}| - |U_{(i,j) \in I', P_{ij}}|.$$

But from Theorem 3.4, this equals  $(n^m - |U_{(i,j) \in I', P_{ij}}|)/n$ .

Hence  $|(U_{(i,j) \in I', P_{ij}}) \cap P_{i_0 j_0}| = |P_{i_0 j_0}| - (n^m - |U_{(i,j) \in I', P_{ij}}|)/n$ . Q.E.D.

Consider now a case when  $|I_{i_0}| > 1$ .

Theorem 3.5 Given  $\{P_{ij}\}_{(i,j) \in I'}$ ,  $I' \subseteq I$ , and  $\{P_{ij}\}_{(i,j) \in I_{i_0}}$  where

$I_{i_0} \subseteq I$ ,  $|I_{i_0}| > 1$ , and  $i_0 \notin D_{I'}$ . Furthermore there exists

$(i', j') \in I'$  with  $i' = j_1$  and  $j' = j_2$  for some  $(i_0, j_1)$  and

$(i_0, j_2) \in I_{i_0}$ . Then  $|(U_{(i,j) \in I_{i_0}, P_{ij}}) \cap (U_{(i,j) \in I', P_{ij}})| =$

$$|U_{(i,j) \in I_{i_0}, P_{ij}}|.$$



Proof: Clearly  $(\cap_{(i,j) \in I_0} P_{ij}) \cap (U_{(i,j) \in I'} P_{ij}) \subseteq \cap_{(i,j) \in I_0} P_{ij}$ .

Then there exists  $(i', j') \in I'$  with  $i' = j_1$  and  $j' = j_2$  for some  $(i_0, j_1)$  and  $(i_0, j_2) \in I_0$ . If  $p \in P_{i_0 j_1} \cap P_{i_0 j_2}$  then the path  $p$  passes through

vertices  $x_{i_0 k'}$ ,  $x_{j_1 k''}$ , and  $x_{j_2 k''}$  where  $k' = k''$  and  $k'' = k'''$ . Hence

$k' = k'''$  and therefore  $p \in P_{j_1 j_2} = P_{i' j'} \subseteq U_{(i,j) \in I'} P_{ij}$  since  $(i', j') \in I'$ .

But  $\cap_{(i,j) \in I_0} P_{ij} \subseteq P_{i_0 j_1} \cap P_{i_0 j_2}$ . Therefore  $\cap_{(i,j) \in I_0} P_{ij} \subseteq$

$(\cap_{(i,j) \in I_0} P_{ij}) \cap (U_{(i,j) \in I'} P_{ij})$ . Hence

$|(\cap_{(i,j) \in I_0} P_{ij}) \cap (U_{(i,j) \in I'} P_{ij})| = |\cap_{(i,j) \in I_0} P_{ij}|$ . Q.E.D.

Corollary 3.3 Given  $|X| = |(U_{(i,j) \in I'} P_{ij}) \cup (U_{(i,j) \in I''} P_{ij})|$

where  $I', I'' \subseteq I$  such that

1. for any  $(i'', j'') \in I''$  then  $i'' = i_0$ , and
2. for any  $(i', j') \in I'$  and  $(i'', j'') \in I''$  then  $i' < i''$ .

Then  $|X \cup P_{i_0 j_0}| = |X| + (n^m - |U_{(i,j) \in I'} P_{ij}|)/n$ .

Proof:  $|X \cup P_{i_0 j_0}| = |(U_{(i,j) \in I'} P_{ij}) \cup (U_{(i,j) \in I''} P_{ij})|$   
 $+ |P_{i_0 j_0}| - |P_{i_0 j_0} \cap (U_{(i,j) \in I''} P_{ij})|$   
 $- \{|P_{i_0 j_0} \cap (U_{(i,j) \in I'} P_{ij})|$   
 $- |P_{i_0 j_0} \cap (U_{(i,j) \in I'} P_{ij}) \cap (U_{(i,j) \in I''} P_{ij})|\}.$

But from Theorem 3.5,  $|P_{i_0 j_0} \cap (U_{(i,j) \in I''} P_{ij})| =$

$|P_{i_0 j_0} \cap (U_{(i,j) \in I'} P_{ij}) \cap (U_{(i,j) \in I''} P_{ij})|.$

Thus  $|X \cup P_{i_0 j_0}| = |X| + |P_{i_0 j_0}| - |P_{i_0 j_0} \cap (U_{(i,j) \in I'} P_{ij})|.$



From Corollary 3.2,  $|X \cup P_{i_0 j_0}| = |X| + (n^m - |U_{(i,j) \in I, P_{ij}}|)/n$ . Q.E.D.

Given an adjacency matrix  $A$  of a graph  $G$ . The matrix  $A$  will be said to have the triangle property if for any elements  $a_{i_1 j_1}, a_{i_1 j_2} = 1$  where  $i_1 > j_1 > j_2$ , then  $a_{j_1 j_2} = 1$ . To illustrate what it means for a matrix to have the triangle property, consider a symmetric boolean matrix  $A$  where

$$A = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & a_{j_1 j_2} & \\ & & a_{i_1 j_2} & a_{i_1 j_1} & \ddots & 0 \end{bmatrix}.$$

The triangle property implies that whenever two non-zero elements  $a_{i_1 j_2}$  and  $a_{i_1 j_1}$  lie below the main diagonal of symmetric matrix  $A$  and in the same row, then the element in the  $(j_1, j_2)$  position of  $A$  is non-zero.

From Theorems 3.4 and 3.5 and Corollary 3.3, the triangle property is sufficient for the order of computation of  $|U_{(i,j) \in I, P_{ij}}|$  to be bounded by a polynomial. In fact, the order of computation is bounded by  $|E|$ .

The number of steps that can be computed in a second with current computers is in the order of  $10^6$ . Then  $|E|$ , the number of edges in a graph, is certainly a feasible number of steps to compute, even for very large graphs.

### 3.3.4 Almost Algebraic Cases

The method presented in 3.3.2 requires an order of computation that is generally exponential. In the previous section properties of the adjacency matrix which are sufficient for the order of computation to be



algebraic (bounded by a polynomial) were given. Cases for which the order of computation is 'almost' algebraic or equivalently 'slightly' exponential will now be discussed.

The first case occurs when the adjacency matrix  $A$  of a graph  $G(V,E)$  has the form

$$A = \begin{bmatrix} & & \\ & A_1 & \\ & & \end{bmatrix}$$

and the following properties:

1. Submatrix  $A_1$  need not have the triangle property.
2. For any  $a_{i_1 j_1}, a_{i_1 j_2} \in A$  and  $\notin A_1$ ,  $a_{i_1 j_1} = a_{i_1 j_2} = 1$ ,

and  $i_1 > j_1 > j_2$ , then  $a_{j_1 j_2} = 1$ .

Depending on the number of non-zero elements in  $A_1$ , the number of steps required to compute  $|U_{(i,j) \in I} P_{ij}|$  can be quite feasible even though  $|V|$  and/or  $|E|$  is very large. More precisely, the order of computation is exponential to a degree depending on the number of non-zero elements in the submatrix  $A_1$ .

Suppose now that the adjacency matrix  $A$  of a graph  $G(V,E)$  with  $|V| = m$  has the triangle property except that  $a_{j_2 j_1} = 0$  for some  $a_{i_1 j_1} = a_{i_1 j_2} = 1$  in  $A$ ,  $j_1' > j_2'$ . Suppose during the determination of  $|U_{(i,j) \in I} P_{ij}|$  that  $|U_{(i,j) \in I} P_{ij}|$  has been evaluated for some  $I' \subseteq I$  ( $I$  is as defined earlier) and that  $|(U_{(i,j) \in I} P_{ij}) \cup (P_{i_0 j_0})|$  is to be determined. The results of



the previous section apply except when  $(i'_1, j'_2) \in I'$  and  $(i_0, j_0) = (i'_1, j'_1)$ .

Define  $I'' \subseteq I'$  as

$$I'' = \{(i, j) \mid (i, j) \in I' \text{ and } i \leq j_2\}$$

and write the elements of  $I' - I'' (\neq \emptyset)$  as  $(i_1, j_1), \dots, (i_K, j_K)$ . Then

$$\begin{aligned} |(U_{(i,j) \in I', P_{ij}}) \cup (P_{i_0 j_0})| &= |U_{(i,j) \in I', P_{ij}}| + |P_{i_0 j_0}| \\ &\quad - |P_{i_0 j_0} \cap (U_{(i,j) \in I'', P_{ij}})| \\ &= \{|P_{i_0 j_0} \cap P_{i_1 j_1}| - \\ &\quad |P_{i_0 j_0} \cap P_{i_1 j_1} \cap (U_{(i,j) \in I'', P_{ij}})| - \\ &\quad \vdots \\ &\quad |P_{i_0 j_0} \cap P_{i_1 j_1} \cap \dots \cap P_{i_K j_K} \cap \\ &\quad (U_{(i,j) \in I'', P_{ij}})| \dots\}. \end{aligned}$$

Theorem 3.5 applies to all the terms in the above expression except the one corresponding to  $|P_{i_1 j_2} \cap P_{i_1 j_1} \cap (U_{(i,j) \in I'', P_{ij}})|$ . In order to evaluate such a term the following theorem is proved.

**Theorem 3.6** Given  $I_1$  and  $I_2 \subseteq I$  where for any  $(i_1, j_1)$  and  $(i_2, j_2) \in I_2$

then  $i_1 = i_2$ . Also  $I_1$  and  $I_2$  are such that for any  $(i_1, j_1) \in I_1$  and  $(i_2, j_2) \in I_2$  then  $i_1 \leq j_2$ . Then

$$|(U_{(i,j) \in I_2, P_{ij}}) \cap (U_{(i,j) \in I_1, P_{ij}})| = (|U_{(i,j) \in I_1, P_{ij}}|) / n^{|I_2|}.$$



Proof: Define the function  $f(i,j) = i$  and let  $\max_{(i,j) \in I_1} \{f(i,j)\} = i$ . Let  $x_{ij}$  be a vertex in the tree  $T$  such that there exists  $p \in \bigcup_{(i,j) \in I_1} P_{ij}$  passing through  $x_{ij}$ ,  $0_{x_{ij}}$  be the number of distinct elements  $p \in \bigcup_{(i,j) \in I_1} P_{ij}$  passing through  $x_{ij}$ , and  $Q = \{x_{ij}\}$  be the set of all such vertices. Then

$$\left| \bigcup_{(i,j) \in I_1} P_{ij} \right| = \sum_Q 0_{x_{ij}}.$$

Furthermore,  $x_{ij}$  is the root of the subtree with  $n^{m-i} = 0_{x_{ij}}$  terminal nodes in  $T$ . From Lemmas 3.3 and 3.5,  $|\bigcap_{(i,j) \in I_2} P_{ij}| = 0_{x_{ij}}/n^{|I_2|}$  in each subtree. Therefore  $|\bigcap_{(i,j) \in I_2} P_{ij} \cap \bigcup_{(i,j) \in I_1} P_{ij}| = \sum_Q 0_{x_{ij}}/n^{|I_2|} = |\bigcup_{(i,j) \in I_1} P_{ij}|/n^{|I_2|}$ . Q.E.D.

From Theorem 3.6, not only is the evaluation of

$$|P_{i_1 j_1} \cap P_{i_1 j_2} \cap \bigcup_{(i,j) \in I_1} P_{ij}|$$

immediate but a generalization of the above holds when the adjacency matrix  $A$  is such that  $A$  has the triangle property except if  $a_{j_k j_{k'}} = 0$  for  $a_{i_1 j_1} = \dots = a_{i_1 j_k} = 1$  in some row in  $A$  ( $j_k > j_{k+1}$  for any  $k$ ,  $1 \leq k \leq K-1$ ) for all  $k, k'$  in the interval  $[1, K]$ .

The computation is exponential to a degree depending on the number of non-zero elements  $a_{ij}$ ,  $i > j$ , in rows  $j_K + 1, \dots, i_1$ .

Clearly the above applies to such conditions existing in more than one row of the adjacency matrix  $A$ .



### 3.3.5 An Example for Comparison to Previously Known Methods

A graph for which previously known methods for the determination for the existence of an  $n$ -coloration are indeterminate is illustrated below.

Consider a connected graph  $G$  with vertex set  $V$ ,  $|V| = 17$ , and an adjacency matrix  $A$  where

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} v_1 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ v_{17} \end{matrix}$$

Suppose that the existence of a 3-coloration of  $G$  is to be determined.

The existence of a 3-coloration cannot be determined from the known

bounds of  $\chi(G)$ .  $K(G)$ , the order of the largest complete subgraph,

is the best known lower bound of  $\chi(G)$ . Using an exhaustive search,

$K(G)$  is found to equal 3. The degree of each vertex  $v_i$  in  $G$  equals

$d_i = \sum_{j=1}^{17} a_{ij}$ . By comparison of the sums  $d_i$ , the maximal degree  $D$ ,

the upper bound of Brooks [11], is found to equal 8. In order to

evaluate  $\alpha(G)$ , the upper bound of Welsh and Powell [46], reorder the sums

$\sum_{j=1}^{17} a_{ij}$ ,  $i=1, \dots, 17$  according to decreasing order and obtain the

sequence 8, 8, 6, 4, 4, 4, 4, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2. To evaluate  $\alpha(G)$ ,

find the minimum of the value,



increased by 1, of the  $i$ th element in the above sequence and of  $i$ . Doing this for each element  $i$  ( $i=1, \dots, 17$ ) gives rise to the sequence 1,2,3,4, 4,4,4,3,3,3,3,3,2,2,2,2,2. Then  $\alpha(G)$ , which equals the maximal element in the sequence just obtained, equals 4.

To find the upper bound provided by the method of Peck and Williams [41], evaluate the degree  $d_i = \sum_{j=1}^{17} a_{ij}$  for each vertex  $v_i$ ,  $i=1, \dots, 17$  to obtain  $d_1, \dots, d_{17}$  as 4,3,8,3,6,2,2,2,8,3,3,4,3,4,2,3,2. Now assign as many vertices as possible, beginning with the vertex of maximal degree and always choosing the vertex with as high degree possible first, to the first color  $c_1$ . Remove those vertices assigned to the first color from the set of all vertices and then repeat the same procedure, assigning vertices to the second color  $c_2$ . Continue on this basis until all vertices have been assigned then resulting coloration is as follows: Vertices  $v_3, v_{12}, v_{14}, v_{13}$ , are assigned to color  $c_1$ , vertices  $v_9, v_1, v_{16}, v_6, v_7$  are assigned to color  $c_2$ , vertices  $v_5, v_4, v_{11}, v_8, v_{15}, v_{17}$  are assigned to color  $c_3$ , and vertices  $v_2, v_{10}$  are assigned to color  $c_4$ . Since 4 colors have been used, the upper bound provided equals 4.

The method of Williams [47] is very similar to that of Peck and Williams except that instead of using the sequence of degrees  $d_1, \dots, d_{17}$  for determining the order in which to choose the vertices, use a sequence  $d_1^s, \dots, d_{17}^s$  where

$$d_i^s = \sum_{j=1}^{17} a_{ij} d_j^{s-1},$$

$$d_j^1 = d_j, \text{ and}$$

$s$  an integer in the order of  $\sqrt[3]{17} (\approx 3)$ .



The sequence  $d_1^3, \dots, d_{17}^3$  is easily shown to be 76,90,162,60,119,60,46,49, 154,76,80,73,47,75,48,43,48. Then the resulting coloration is as follows: Vertices  $v_3, v_{10}, v_{14}, v_6, v_{15}$ , are assigned to color  $c_1$ , vertices  $v_9, v_2, v_4, v_8, v_{13}$ , are assigned to color  $c_2$ , vertices  $v_5, v_{11}, v_1, v_{12}, v_7, v_{17}$ , are assigned to color  $c_3$ , and vertex  $v_{16}$  is assigned to color  $c_4$ . Again, since 4 colors have been used, the upper bound provided equals 4.

The method of Formby [20] arbitrarily assigns each vertex a different color and then attempts to reduce the number of colors required. Applying it to the above example, assign vertices  $v_1, \dots, v_{17}$  to colors  $c_1, \dots, c_{17}$  respectively. Then find the color with the lowest possible subscript that can replace color  $c_1$ . Next find the color with the lowest possible subscript that can replace color  $c_2$ . Continue in this manner until all colors have been considered. Any colors that cannot be replaced must be a necessary color. For the above example, this method provides a coloration of  $G$  that requires 4 colors and hence provides an upper bound equal to 4.

Thus the existence of a 3-coloration of  $G$  cannot be determined from the bounds of  $\chi(G)$  as 3 is in the interval  $[B_L, B_U)$  where  $B_L = 3$  and the best upper bound  $B_U$  is equal to 4.

Now pose the determination of the existence of a 3-coloration for the graph  $G$  as a multi-stage decision problem where at each stage  $i$ , vertex  $v_i$  must be assigned to one of three colors such that  $v_i$  is not assigned to the same color as some  $v_j$ ,  $j < i$ , that is adjacent to  $v_i$ . Previous discussion showed that this approach is generally not feasible



in that the computation required often approaches that of total enumeration of all possible 3-colorations. A further demonstration thereof is given by applying this approach to the example being considered.

The existence of a 3-coloration is established after a decision has been completed at each stage. The non-existence of a 3-coloration is established only after all possible decisions have been attempted.

Suppose the decisions at stages 1,...,15 have been completed and let these decisions be as follows where  $(v_i, c_j)$  indicates vertex  $v_i$  is assigned to color  $c_j$ :  $(v_1, c_1)$ ,  $(v_2, c_2)$ ,  $(v_3, c_3)$ ,  $(v_4, c_2)$ ,  $(v_5, c_1)$ ,  $(v_6, c_3)$ ,  $(v_7, c_1)$ ,  $(v_8, c_2)$ ,  $(v_9, c_2)$ ,  $(v_{10}, c_3)$ ,  $(v_{11}, c_1)$ ,  $(v_{12}, c_1)$ ,  $(v_{13}, c_2)$ ,  $(v_{14}, c_3)$ , and  $(v_{15}, c_3)$ . But now no decision is possible at stage 16 as vertex  $v_{16}$  is adjacent to vertices  $v_{12}$ ,  $v_{13}$ , and  $v_{14}$ . Hence back-up, so as to alter previous decisions, is required to determine the existence of a 3-coloration. Furthermore, the order of computation required approaches total enumeration of all solutions unless of course a 3-coloration is found prior to this exhaustive search.

To determine the existence of a 3-coloration of  $G$  using the results presented in the previous sections then  $|U_{(i,j) \in I} P_{ij}|$  must be computed where, corresponding to the adjacency matrix  $A$ ,

$$I = \{(2,1), (3,1), \dots, (12,10), (13,5), \dots, (15,12), (16,13), \dots, (17,14)\}.$$

Let  $A_1$  equal the submatrix that is indicated within the matrix  $A$ . Thus  $A$  has the form corresponding to an almost algebraic case in Section 3.4.4. That is matrix  $A$  has the triangle property, except for the submatrix  $A_1$ .



Hence to determine  $|U_{(i,j) \in \bar{I}} P_{ij}|$  where  $\bar{I} = \{(2,1), \dots, (15,12)\} \subseteq I$ , the results of Sections 3.4.2 and 3.4.3 may be employed.

From Lemma 3.3,  $|P_{21}| = 3^{17-1}$ . Let  $y_1 = 3^{17-1}$ . Then from Theorem 3.4,

$$\begin{aligned} |P_{21} \cup P_{31}| &= |P_{21}| + (3^{17} - |P_{21}|)/3 \\ &= y_1 + (3^{17} - y_1)/3 \end{aligned}$$

Let  $y_2 = |P_{21} \cup P_{31}|$ .

Now let  $I' = \{(2,1)\}$  and  $I'' = \{(3,1)\}$ . Then from Corollary 3.3,

$$\begin{aligned} |P_{21} \cup P_{31} \cup P_{32}| &= |(U_{(i,j) \in I'} P_{ij}) \cup (U_{(i,j) \in I''} P_{ij}) \cup P_{32}| \\ &= y_2 + (3^{17} - y_2)/3. \end{aligned}$$

Let  $y_3 = y_2 + (3^{17} - y_2)/3$ .

Continuing in this manner then

$$|U_{(i,j) \in \bar{I}} P_{ij}| = 3^3 \cdot (3^{14} - 2).$$

Let  $y_{26} = |U_{(i,j) \in \bar{I}} P_{ij}|$ .

Next  $|(U_{(i,j) \in \bar{I}} P_{ij}) \cup P_{16\ 12}| = y_{26} + (3^{17} - y_{26})/3$ .

Let  $y_{27} = y_{26} + (3^{17} - y_{26})/3$ .

To evaluate  $|(U_{(i,j) \in \bar{I}} P_{ij}) \cup P_{16\ 12} \cup P_{16\ 13}|$ , the results of Section 3.3.4 must be employed.



As in Section 3.3.4, define

$$I'' = \{(2,1), \dots, (12,10)\}, \text{ and}$$

$$I' = \{(2,1), \dots, (16,12)\}.$$

$$\begin{aligned} \text{Then } I' - I'' &= \{(i_1, j_1), \dots, (i_K, j_K)\} \\ &= \{(13,5), \dots, (16,12)\}. \end{aligned}$$

$$\text{Clearly } |U_{(i,j) \in I''} P_{ij}| = \gamma_{20}$$

From the discussion in Section 3.3.4, then

$$\begin{aligned} |(U_{(i,j) \in I'} P_{ij}) \cup P_{16 \ 13}| &= |U_{(i,j) \in I'} P_{ij}| + \{|P_{16 \ 13}| - \\ &\quad |P_{16 \ 13} \cap (U_{(i,j) \in I'} P_{ij})|\} \\ &= \{|P_{16 \ 13} \cap P_{13 \ 5}| - |P_{16 \ 13} \\ &\quad \cap P_{13 \ 5} \cap (U_{(i,j) \in I'} P_{ij})|\}. \end{aligned}$$

$$\text{Let } \gamma_{29} = |(U_{(i,j) \in I'} P_{ij}) \cup P_{16 \ 13}|.$$

Using Theorems 3.3 and 3.6,

$$\gamma_{29} = \gamma_{28} + \{(3^{17} - \gamma_{28})/3\} - \{3^{15} - \gamma_{20}/3^2\}.$$

Thus  $|U_{(i,j) \in I'} P_{ij}|$  can be evaluated and found to be equal to 6 for graph  $G$ . But  $|U_{(i,j) \in I'} P_{ij}| = 6 < 3^{17}$ . Thus a 3-coloration of  $G$  does exist.

To realize the advantage of the given results compared to using the multi-stage decision process, consider a general graph  $G'$  with  $|V| = m$  and adjacency matrix  $A'$  of the form where







computation required for determination of the existence of an  $n$ -coloration. This section first gives a relationship between the triangle property and triangulated graphs. Secondly, the order of the largest complete subgraph contained in the medial graph of any cubic graph is shown to be 3 by means of being able to determine the 'extent' to which the triangle property exists within a sequence of adjacency matrices. The significance of this result is given.

### 3.4.1 The Relationship of the Triangle Property to Triangulated Graphs

Before relating the adjacency matrix of a graph having the triangle property to known results in graph theory, a few definitions are required. Namely, a chord is an edge joining two non-consecutive vertices of a cycle in a graph. A graph is said to be triangulated if every cycle of length greater than 3 possesses a chord.

Then the following theorem can be proved.

**Theorem 3.7** If a graph  $G$  has an adjacency matrix with the triangle property then  $G$  is a triangulated graph.

**Proof:** The result is immediate for a graph  $G(V,E)$  with  $|V| \leq 3$ . The proof follows from induction on  $m = |V|$ . Suppose it holds for graphs with  $m$  vertices. Consider now a graph with  $m + 1$  vertices whose adjacency matrix  $A$  has the triangle property. The submatrix  $A'$  of  $A$  consisting of the first  $m$  rows (columns) of  $A$  corresponds to a subgraph  $G'$  of  $G$  with  $m$  vertices and an adjacency matrix  $A'$  that has the triangle property. Thus  $G'$  is a triangulated subgraph.  $G'$  corresponds to a graph  $G$  with vertex  $v_{m+1}$  and edges incident to  $v_{m+1}$  deleted. From the triangle property of  $A$  then for any pair of vertices adjacent to vertex  $v_{m+1}$ , there exists an edge



joining that pair of vertices. Hence any cycle passing through  $v_{m+1}$  has a chord. Thus all cycles in  $G$  have a chord. Q.E.D.

In order to prove a converse to the statement in Theorem 3.8, the following lemmas are required.

**Lemma 3.6** Given a triangulated graph  $G(V, E)$ . Let  $\bar{v} \in V$  and  $E_{\bar{v}} = \{e \mid e \in E \text{ and } e \text{ incident to } \bar{v}\}$ . Then the subgraph  $G'(V - \bar{v}, E - E_{\bar{v}})$  of  $G$  is triangulated.

**Proof:** Since any cycle in  $G$  not passing through  $\bar{v}$  has a chord that does not coincide to any edge in  $E_{\bar{v}}$ , therefore  $G'$  is triangulated. Q.E.D.

**Lemma 3.7** If the graph  $G(V, E)$  has a cycle, then there exists a  $\bar{v} \in V$  such that either  $d(\bar{v}) = 1$  or such that there exists a cycle in  $G$  passing through any pair of distinct edges incident to  $\bar{v}$ .

**Proof:** Choose any  $v = v_1 \in V$ . If either of the required properties hold for  $v_1$  then the proof is complete. Otherwise there exists a vertex  $v = v_2$  adjacent to  $v_1$  such that there does not exist a cycle in  $G$  with edge  $(v_1, v_2)$ . If  $d(v_2) = 1$  then the proof is complete. Otherwise  $d(v_2) \geq 2$ . If  $v_2$  is adjacent to a vertex  $v = v_3$ ,  $v_3 \neq v_1$ , for which either of the properties hold, then the proof is complete. If not, there exists a  $v = v_3$  such that  $v_3$  is adjacent to  $v_2$  and for which neither of the desired properties hold. Now  $v_3 \neq v_1$  for otherwise there exists a cycle with edge  $(v_1, v_2)$  in  $G$ . But this is a contradiction, the implication being that there exists a vertex adjacent to  $v_2$  such that it has one of the desired properties. Furthermore, if there exists a



cycle in  $G$  with edge  $(v_2, v_3)$  then choose a vertex  $v = v_4$  such that there does not exist a cycle with edge  $(v_3, v_4)$ . Such a vertex exists by definition of  $v_3$ . Also  $v_4 \neq v_2$ . If  $v_4 = v_1$  then there is a contradiction since edge  $(v_1, v_2)$  is in  $G$ .

Replace  $v_1$  by  $v_3$  (by  $v_4$  if there exists a cycle in  $G$  with edge  $(v_2, v_3)$ ),  $v_2$  by  $v_4$  ( $v_5$ ),  $v_3$  by  $v_5$  ( $v_6$ ), and  $v_4$  by  $v_7$ , if necessary, in the above. Then repeat the procedure.

Each iteration of the above procedure adds 2 (3) edges to a path in  $G$ . After at most  $m / 2$  or  $(m + 1) / 2$  iterations, according as  $m + 1$  is odd or even, a vertex with the desired characteristics will have been determined. Otherwise a cycle containing an edge which presumably was not in any cycle of  $G$  has been determined and there exists a contradiction. Q.E.D.

The next theorem proves the converse to Theorem 3.7.

**Theorem 3.8** For any triangulated graph  $G(V, E)$  there exists a labelling of the vertices of  $G$  such that the adjacency matrix has the triangle property.

**Proof:** The result holds for triangulated graphs with  $|V| = m \leq 3$  vertices. Assume it holds for all triangulated graphs with  $\leq m$  vertices.

Consider any triangulated graph  $G$  with  $m + 1$  vertices. If there does not exist a cycle of length  $\geq 3$  in  $G$  then  $G$  is a tree and must have a vertex  $\bar{v}$  with degree 1. Let  $E_{\bar{v}}$  be as previously defined. Then from Lemma 3.6 the subgraph  $G'(V - \bar{v}, E - E_{\bar{v}})$  of  $G$  is a triangulated subgraph.



Hence  $G'$  has a labelling such that the corresponding adjacency matrix has the triangle property. Label the vertices in  $V - \bar{v}$  of  $G$  as in  $G'$  and  $\bar{v}$  as  $v_{m+1}$ . Then since  $d(\bar{v}) = 1$ , the proof is complete.

Now suppose there exists a cycle of length  $\geq 3$  in  $G$ . From Lemma 3.7, there exists a vertex  $\bar{v} \in V$  such that  $d(\bar{v}) = 1$  or such that there exists a cycle passing through any pair of edges incident to  $\bar{v}$ . If  $\bar{v}$  is such that  $d(\bar{v}) = 1$  then proceed similarly as in the case for  $G$  having no cycle of length  $\geq 3$ . If there exists a vertex  $\bar{v} \in V$  such that there exists a cycle passing through any pair of edges incident to  $\bar{v}$ , let  $c_1$  be any cycle passing through  $\bar{v}$ . Let  $c_1$  have length  $L_1$ . Suppose  $L_1 > 3$ . Since  $G$  is triangulated there exists a cycle  $c_2$  of length  $L_2 < L_1$  and passing through  $\bar{v}$ . Similarly if  $L_2 > 3$ , there exists a cycle  $c_3$  of length  $L_3 < L_2$  and passing through  $\bar{v}$ . Continue in this fashion until a cycle  $c_i$  of length  $L_i = 3$  and passing through  $\bar{v}$  is obtained. This holds for any pair of distinct edges incident to  $\bar{v}$ . Hence any two vertices adjacent to  $\bar{v}$  are themselves adjacent.

Again  $G'(V - \bar{v}, E - E_{\bar{v}})$  has a labelling such that the corresponding adjacency matrix has the triangle property. Then label the vertices of  $G$  in  $V - \bar{v}$  as for  $G'$  and label  $\bar{v}$  as  $v_{m+1}$ . Then from the above arguments, there is an adjacency matrix of  $A$  with the triangle property. Q.E.D.

The relationship between the triangle property and triangulated graphs provides an alternative method for identifying a graph  $G$  as being triangulated: Rather than determine whether every cycle of length greater than 3 possesses a chord, establish whether  $G$  has an adjacency matrix with the triangle property.



### 3.4.2 Maximal Complete Subgraphs Within Certain Medial Graphs

Prior to stating the main theorem proved in this section, cubic and medial graphs are defined. As well, some results that pertain to cubic graphs are stated. They are required in the proof of the main theorem.

A cubic graph is a regular graph of degree 3. The medial graph  $\bar{G}$  of a given graph was defined by Ore [40] in the following manner. On each edge  $e_i \in E$  of  $G$ , a midpoint  $\bar{v}_i$  is selected. When two edges  $e_i$  and  $e_j$  in  $G$  are incident with the same vertex, join  $\bar{v}_i$  and  $\bar{v}_j$  by an edge  $(\bar{v}_i, \bar{v}_j)$ . This procedure yields a new graph  $\bar{G}$  with vertices  $\bar{v}_i \in V$  and edges  $(\bar{v}_i, \bar{v}_j) \in \bar{E}$ .  $\bar{G}$  is called the medial graph of  $G$ .

There exist cubic graphs  $G(V, E)$  of even order  $V$  for  $|V| = 4$ . Let  $|V| = R$ . For  $R = 4$ , there is only one such graph, namely the complete graph  $K_4$ . For  $R = 6$  there are two of them, the 6 vertex Kuratowski graph (Figure 3.5) and the graph  $G$  in Figure 3.6.

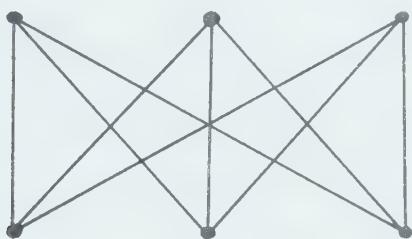


Figure 3.5  
Kuratowski Graph

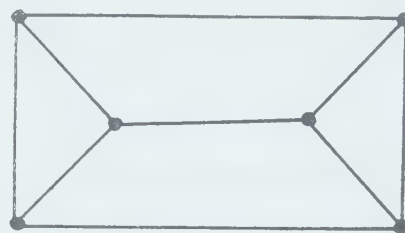


Figure 3.6  
Graph  $G$

Denote an edge  $e$  joining vertices  $v$  and  $v'$  by  $e = (v, v')$ . Let  $e_1 = (v_2, v_4)$  and  $e_2 = (v_3, v_5)$  be two edges in  $G$  where the vertices  $v_2$ ,  $v_3$ ,  $v_4$ , and  $v_5$  are all distinct. The  $H$ -expansion (see [40]) of  $G$  with respect to  $e_1$  and  $e_2$  is obtained by eliminating  $e_1$  and  $e_2$  and adjoining two vertices  $v_0$  and  $v_1$  with edges



$$(v_0, v_1)(v_0, v_2)(v_0, v_5)(v_1, v_3)(v_1, v_4)$$

or also by

$$(v_0, v_1)(v_0, v_2)(v_0, v_3)(v_1, v_4)(v_1, v_5).$$

There exists a successive construction of cubic graphs as a result of the following theorem due to Johnson (see [40]).

**Theorem 3.9** For  $R \geq 6$ , every connected cubic graph on  $R + 2$  vertices is an H-expansion of a connected cubic graph on  $R$  vertices.

Hence any cubic graph of order  $R > 6$  can be obtained from a cubic graph of order 6 with a sequence of  $(R - 6)/2$  H-expansions. Each H-expansion introduces 2 new vertices and an additional 3 edges.

If the vertices of a graph are appropriately indexed and if  $G$  contains a complete subgraph  $K_k$ , then the adjacency matrix  $A$  corresponding to  $G$  contains a  $k \times k$  submatrix whose non-diagonal elements are all equal to one. The same effect is produced by an arbitrary indexing of the vertices followed by rearranging rows and columns by a sequence of operations, each interchanging a pair  $(i, j)$  of rows and a pair  $(i, j)$  of columns. Conversely, if an adjacency matrix of a graph  $G$  contains  $k \times k$  submatrix whose properties are as above (or equivalently, if such a submatrix can be obtained by rearrangement of rows and columns of  $A$ ), then  $G$  contains a complete subgraph  $K_k$ .

**Theorem 3.10** The order of the largest complete subgraph contained in the medial graph of any cubic graph is three.

**Proof:** For cubic graphs of order equal to 6 this is immediate from Figures 3.5 and 3.6.



A cubic graph  $G$  can be constructed in a series of H-expansions, at each stage  $i$  of which there is a cubic graph  $G_i$ . Corresponding to each  $G_i$  and  $G$  are medial graphs  $\bar{G}_i$  and  $\bar{G}$  respectively. Let the initial cubic graph of order 6 be  $G_0$ . Suppose that the order of the largest complete subgraph in  $\bar{G}_i$  is 3. Then it is required to prove that the same holds for  $\bar{G}_{i+1}$ .

Suppose that  $G_{i+1}$  is a H-expansion of  $G_i$  with respect to edges  $e_1 = (v_2, v_4)$  and  $e_2 = (v_3, v_5)$  in  $G_i$ . Denote the order of the medial graph  $\bar{G}_i$  of  $G_i$  by  $l$ . Index the  $l$  vertices of  $\bar{G}_i$  so that the vertices  $\bar{v}_{l-1}$  and  $\bar{v}_l$  correspond to edges  $e_1$  and  $e_2$  of  $\bar{G}_i$  respectively. Designate the two vertices adjoined to  $G_i$  to obtain  $G_{i+1}$  as  $v_0$  and  $v_1$  and the edges adjoined as  $(v_0, v_1)(v_0, v_2)(v_0, v_5)(v_1, v_3)(v_1, v_4)$ . Let the vertices  $\bar{v}_{l-1}$  and  $\bar{v}_l$  of  $\bar{G}_i$  correspond to edges  $(v_0, v_2)$  and  $(v_0, v_5)$  in  $G_{i+1}$ , as well as to vertices  $\bar{v}_{l-1}$  and  $\bar{v}_l$  in  $\bar{G}_{i+1}$  respectively. Let vertices  $\bar{v}_{l+1}$ , and  $\bar{v}_{l+2}$ , and  $\bar{v}_{l+3}$  be the vertices adjoined to  $\bar{G}_i$  to obtain  $\bar{G}_{i+1}$  and correspond to the adjoined edges  $(v_0, v_1)(v_1, v_3)$ , and  $(v_1, v_4)$  in  $G_{i+1}$  respectively. The relevant vertices of  $G_i$  and  $\bar{G}_i$  are indicated in Figure 3.7. Similarly for  $G_{i+1}$  and  $\bar{G}_{i+1}$  in Figure 3.8.

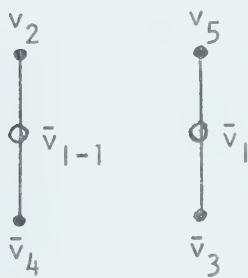


Figure 3.7  
Subgraph of  $G_i$

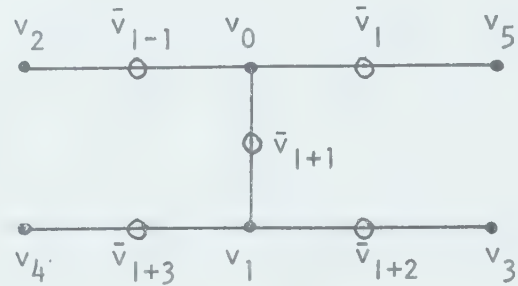


Figure 3.8  
Subgraph of  $G_{i+1}$



If  $\bar{A}_{i+1}$  is the adjacency matrix corresponding to  $\bar{G}_{i+1}$ , then by construction (see Figure 4.4), the elements in positions  $(l+3, l+2)$ ,  $(l+3, l+1)$ ,  $(l+2, l+1)$ ,  $(l+1, l)$ ,  $(l+1, l-1)$ , and  $(l, l-1)$  are all equal to 1. The vertices  $v_2, v_3, v_4$ , and  $v_5$  are all distinct. Thus the elements of  $\bar{A}_{i+1}$  in positions  $(l+2, l-1)$  and  $(l+2, l)$  equal 0. The elements in positions  $(l+1, l), \dots, (l+1, l-2)$  of  $\bar{A}_{i+1}$  are all zero. Then

$$\bar{A}_{i+1} = \begin{bmatrix} 0 & & & & & & \\ & & & & & & \\ & & & & & & \\ & & 0 & & & & \\ & & 1 & 0 & & & \\ & 0 & 1 & 1 & 0 & & \\ & & 0 & 0 & 1 & 0 & \\ & & & & 1 & 1 & 0 \end{bmatrix} \begin{matrix} \bar{v}_1 \\ \\ \\ \\ \bar{v}_{l+3} \end{matrix}$$

with the determinant elements as indicated. Since  $\bar{A}_{i+1}$  is symmetric, only elements below the diagonal are given.

Hold the  $(l+3)^{\text{rd}}$  and  $(l+2)^{\text{nd}}$  rows (columns) of  $\bar{A}_{i+1}$  fixed. Then the elements which equal 0 in the  $(l+1)^{\text{st}}$  and  $(l+2)^{\text{nd}}$  rows of  $\bar{A}_{i+1}$  make it impossible to rearrange the first  $l+1$  rows and columns so as to obtain a  $4 \times 4$  submatrix corresponding to a complete subgraph  $K_4$  whose set of vertices includes at least one of  $\bar{v}_{l+1}$ ,  $\bar{v}_{l+2}$ , or  $\bar{v}_{l+3}$ .

Since  $\bar{G}_{i+1}$  is regular of degree 4, then any complete subgraph  $K_4$  containing  $\bar{v}_l$  must necessarily contain vertex  $\bar{v}_{l+1}$ . Hence it is impossible for vertex  $\bar{v}_l$  to be a vertex in a complete subgraph  $K_4$ .

The existence of complete subgraphs  $K_4$  in  $\bar{G}_{i+1}$  can now be restricted



to vertices  $\bar{v}_1, \dots, \bar{v}_{l-1}$  for it has been shown that neither of vertices  $\bar{v}_1, \dots, \bar{v}_{l+3}$  can be a vertex on such a subgraph. Let  $\bar{A}_i$  be the adjacency matrix of  $\bar{G}_i$ . Partition  $\bar{A}_{i+1}$  and  $\bar{A}_i$  as follows:

$$\bar{A}_{i+1} = \begin{bmatrix} \begin{bmatrix} A_i' \\ 1 \ 0 \\ 0 \ 1 \ 1 \ 0 \\ 0 \ 0 \ 1 \ 0 \\ 1 \ 1 \ 0 \end{bmatrix} & \begin{bmatrix} \bar{v}_1 \\ \\ \\ \\ \bar{v}_{l+3} \end{bmatrix} \end{bmatrix} \quad \text{and} \quad \bar{A}_i = \begin{bmatrix} \begin{bmatrix} \bar{A}_i' \\ \\ 0 \end{bmatrix} & \begin{bmatrix} \bar{v}_1 \\ \\ \bar{v}_l \end{bmatrix} \end{bmatrix}.$$

$\bar{A}_i'$  differs from  $\bar{A}_i$  in that some non-zero elements of  $\bar{A}_i$  have been set to zero in  $\bar{A}_i'$  as a result of the H-expansion in going from  $G_i$  to  $G_{i+1}$ . Therefore no rearrangement of rows and columns in  $\bar{A}_i'$  will result in a  $4 \times 4$  submatrix whose non-diagonal elements are all non-zero. This holds because  $\bar{G}_i$  contains no complete subgraph of order  $\geq 4$ .

The same results hold when the edges adjoined to  $G_i$  by the H-expansion to obtain  $G_{i+1}$  are  $(v_0, v_1)(v_0, v_2)(v_0, v_3)(v_1, v_4)(v_1, v_5)$ . Q.E.D.

Thus by demonstration of a particular property of an adjacency matrix at each iteration in Johnson's method for construction of an arbitrary cubic graph, it has been shown that the order of the largest complete subgraph contained in the medial graph of any cubic graph is three.

Before being able to state the significance of the above result, colorations of a class of graphs called planar graphs and the medial graph of any cubic graph must be related. However to do so, the following definitions are necessary.



A planar graph is a graph that can be represented in a plane such that the edges do not intersect other than at their endpoints. A graph is said to be edge  $n$ -colorable if each edge can be assigned one of  $n$  (or less) colors in such a way that no two edges with the same color are incident. For planar graphs, a graph is said to be face  $n$ -colorable if each face can be assigned one of  $n$  (or less) colors in such a way that adjoining faces always have different colors. The four color problem is as follows: Are all planar graphs face 4-colorable? Note that adjoining faces of a planar graph  $G$  may be said to correspond to adjacent vertices of a graph  $G'$ . Hence planar graphs may 'correspond' to CTT problems and vice versa.

The following three known theorems relate the face coloration of a planar graph to the edge coloration of a cubic graph.

Theorem 3.11 [40, Theorem 6.3.1, pp 79] A planar graph is face or vertex  $n$ -colorable if and only if its connected components have this property.

Theorem 3.12 [40, Theorem 9.1.1, pp 117] The face  $n$ -coloration of a planar graph can be reduced to the case of cubic graphs.

Theorem 3.13 [40, Theorem 9.3.1, pp 121] A cubic graph is face 4-colorable if and only if it is edge 3-colorable.

But from the construction of the medial graph  $\overline{G}$  of a graph  $G$ ,  $G$  is edge  $n$ -colorable if and only if  $\overline{G}$  is  $n$ -colorable. Thus the four color problem reduces to the existence of a 3-coloration of the medial graph corresponding to an arbitrary cubic graph.



Myers and Liu [37] raise a question concerning the indeterminacy of the relationship between  $\chi(G)$  and  $K(G)$  for face colorations of planar graphs. The above shows that planar graphs reduce to a class of graphs, namely the medial graph of any cubic graph, for which there is no indeterminacy unless the four color conjecture is not true.



## CHAPTER 4

### ADDITIONAL CONSTRAINTS TO CTT PROBLEMS

In the previous chapter, the existence of an  $n$ -coloration of a graph was discussed. However, as seen in Chapter 2, the edges of a graph represent only those constraints in CTT problems that prevent assignment of pairs of classes to the same hour. In this chapter, the existence of an  $n$ -coloration of graphs that have other constraints imposed upon them are considered. Section 4.1 considers those mentioned by Welsh and Powell [46] and already discussed in Chapter 2. These constraints assign given vertices to particular colors. In Section 4.2 discussion is given of those constraints in CTT problems that are equivalent to the prevention of assignment of vertices to certain colors. For both cases, theorems are proved that reduce the existence of an  $n$ -coloration of graphs with these additional constraints imposed upon them to the existence of an  $n$ -coloration of graphs without these additional constraints. In Section 4.3, it is shown how the results of the theorem proved in Section 4.1 may be applied as a basis for more flexible methods of obtaining an  $n$ -coloration of a graph.

#### 4.1 Preassignment Constraints

The constraints that have previously been considered pertain to the prevention of pairs of vertices from being assigned to the same color. Let these be called adjacency constraints. Constraints that preassign vertices to specific colors will be called preassignment constraints. This section describes how a graph with both adjacency and preassignment constraints may be reduced to a graph with only adjacency constraints.



Given  $G(V,E)$  with  $|V| = m$  and a set  $S$  of preassignment constraints. Suppose there exist vertices  $v_i \in V$ ,  $i = 1, \dots, n'$  ( $n' \leq m$ ), that have been preassigned to colors  $c_i$ ,  $i = 1, \dots, n'$ , but there do not exist vertices  $v_i \in V$ ,  $i = 1, \dots, n'+1$  ( $n' + 1 \leq m$ ) that have been preassigned to colors  $c_i$ ,  $i = 1, \dots, n'+1$ . The set  $S$  of preassignments will be said to require  $n'$  colors. An  $n$ -coloration of  $G$  does not exist if  $n' > n$ . It will henceforth be assumed that  $n' \leq n$ .

Consider a graph  $G(V,E)$  with  $|V| = m$  and a set of preassignment constraints and suppose the requirement is to establish the existence of an  $n$ -coloration. Define a  $m \times n$  preassignment matrix  $P_{mn}$  where  $p_{ij} = 1$  if vertex  $v_i \in V$  has been preassigned to color  $c_j$  and  $p_{ij} = 0$  otherwise. For matrix  $P_{mn}$ ,  $\sum_{j=1}^n p_{ij} = 1$  if vertex  $v_i \in V$  has been preassigned to a specified color and  $\sum_{j=1}^n p_{ij} = 0$  otherwise. Denote a graph  $G(V,E)$  with preassignment matrix  $P_{mn}$  by  $G(V,A,P_{mn})$ . If  $P_{mn} \equiv 0$ , write  $G(V,A,P_{mn})$  as  $G(V,A,\emptyset)$ .

Define the sets  $V_j$  where

$$V_j = \{v_i \mid \text{all } v_i \in V \text{ such that } p_{ij} = 1 \text{ where } p_{ij} \in P\}$$

for  $j = 1, \dots, n$ . Clearly  $V_j \cap V_{j'} = \emptyset$  for  $j \neq j'$  and  $V_j = \emptyset$  if there does not exist  $v_i \in V$  such that  $v_i$  is preassigned to color  $c_j$ . The following lemma is immediate.

**Lemma 4.1** A necessary condition for the existence of an  $n$ -coloration of  $G(V,E,P_{mn})$  is that for any  $v_i, v_{i'} \in V_j$ ,  $a_{ii'} = a_{i'i} = 0$  in matrix  $A$ .



Define the graph  $G'$ , corresponding to the graph  $G$ , with vertex set  $V$  as in  $G$  and (symmetric) adjacency matrix  $A'$  with elements  $a'_{ij}$  where

1. if  $a_{ij} \in A$  and  $a_{ij} = 1$ , then  $a'_{ij} = 1$ ,
  2. if  $v_i \in V_k$  and  $v_j \in V_{k'}$ , for some  $k, k'$  where  $k \neq k'$  and  $1 \leq k, k' \leq n$ , then  $a'_{ij} = 1$ , and
  3. if  $v_i, v_{i'} \in V_k$  for some  $k, 1 \leq k \leq n$ , and there exists  $v_j \in V, v_j \notin V_k$ , such that  $a_{i',j} = 1$ , then  $a'_{ij} = 1$ .
- Otherwise  $a'_{ij} = 0$ .

The following theorem proves the desired result.

**Theorem 4.1** A graph  $G(V, A, P_{mn})$  is  $n$ -colorable if and only if  $G'(V, A', \emptyset)$  is  $n$ -colorable.

**Proof:** Suppose  $G(V, A, P_{mn})$  is  $n$ -colorable. Let  $\{C_k\}_{k=1}^n$  be an  $n$ -coloration of  $G$  where  $C_k$  represents the vertices of  $G$  that have been assigned to color  $c_k, k = 1, \dots, n$ . Clearly  $C_k \cap C_{k'} = \emptyset$  for  $k \neq k'$ . For each  $V_{k'}$  such that  $V_{k'} \neq \emptyset, k' = 1, \dots, n$ , there exists exactly one  $k, 1 \leq k \leq n$ , such that  $V_{k'} \subseteq C_k$ . This follows from the definition of  $\{V_{k'}\}_{k'=1}^n$  and  $\{C_k\}_{k=1}^n$ . For any  $k, k', 1 \leq k, k' \leq n$ , there does not exist  $k'', 1 \leq k'' \leq n$ , such that  $V_k \subseteq C_{k''}$  and  $V_{k'} \subseteq C_{k''}$ .

To prove that  $\{C_k\}_{k=1}^n$  is an  $n$ -coloration of  $G'(V, A', \emptyset)$  it is required to show that for any  $v_i, v_j \in C_k, i \neq j$ , then  $a'_{ij} = 0$ . Suppose there exists  $v_i, v_j \in C_k, i \neq j$ , for some  $k, 1 \leq k \leq n$ , such that  $a'_{ij} = 1$ . By definition of  $a'_{ij}$  then either

1.  $a_{ij} = 1$ , or



2.  $v_i \in V_{k'}$  and  $v_j \in V_{k''}$  for some  $k'$  and  $k''$ ,  $k' \neq k''$ , and  $1 \leq k', k'' \leq n$ , or

3. either  $v_i$  or  $v_j \in V_{k_1} \subseteq C_{k_1}$ , say  $v_i$ , and there exists  $v_{i_1} \in V_{i_1}$ ,  $v_{i_1} \neq v_i$ , such that  $a_{i_1, j} = 1$ .

All however contradict that  $\{C_k\}_{k=1}^n$  is an  $n$ -coloration of  $G(V, A, P_{mn})$ .

Thus  $\{C_k\}_{k=1}^n$  is an  $n$ -coloration for  $G'(V, A', \emptyset)$  and hence is  $n$ -colorable.

Now let  $\{C_k\}_{k=1}^n$  be an  $n$ -coloration of  $G'(V, A', \emptyset)$  where  $C_k$  are defined as above. By definition of matrix  $A'$  and the sets  $\{V_k\}_{k=1}^n$ , then for any  $k, k'$ ,  $1 \leq k, k' \leq n$ , there does not exist  $k''$ ,  $1 \leq k'' \leq n$ , such that  $V_k \cap C_{k''} \neq \emptyset$  and  $V_{k'} \cap C_{k''} = \emptyset$ . Suppose  $V_1 \neq \emptyset$ . Choose  $k_1^1$ ,  $1 \leq k_1^1 \leq n$ , such that  $V_1 \cap C_{k_1^1} \neq \emptyset$ . Then unless  $V_1 \subseteq C_{k_1^1}$ , there exists  $k_2^1$ ,  $1 \leq k_2^1 \leq n$ ,  $k_1^1 \neq k_2^1$ , such that  $(V_1 - (V_1 \cap C_{k_1^1})) \cap C_{k_2^1} \neq \emptyset$  because  $V_1 \subseteq \bigcup_{k=1}^n C_k$ . But

$$\{(\{C_k\}_{k=1}^n, k \neq k_1^1, k_2^1), (C_{k_1^1} \cup (V_1 \cap C_{k_2^1})), (C_{k_2^1} - (V_1 \cap C_{k_2^1}))\}$$

is an  $n$ -coloration of  $G'$  for suppose there exists a  $v_i \in V_1 \cap C_{k_2^1}$  and

$v_{i_1} \in C_{k_1^1}$  such that  $a_{i_1, i} = 1$ . Let  $v_{i_1} \in V_1 \cap C_{k_1^1} \neq \emptyset$ . Then by

definition of  $A'$ ,  $a_{i_1, i_1} = 1$ . But  $\{C_k\}_{k=1}^n$  is an  $n$ -coloration of  $G'$ .

Hence there is a contradiction. Thus

$$\{(\{C_k\}_{k=1}^n, k \neq k_1^1, k_2^1), (C_{k_1^1} \cup (V_1 \cap C_{k_2^1})), (C_{k_2^1} - (V_1 \cap C_{k_2^1}))\}$$

is an  $n$ -coloration of  $G'$ . Continue in the above manner until an

$n$ -coloration  $\{C_k^1\}_{k=1}^n$  of  $G'$  with  $V_1 \subseteq C_k^1$  for some  $k$  is obtained. Repeat

the procedure for each  $V_k$ ,  $k = 2, \dots, n$ . The resulting  $n$ -coloration is

also an  $n$ -coloration of  $G$ . Q.E.D.



To illustrate the result in Theorem 4.1, consider a graph  $G(V,E)$  with  $|V| = 8$  and adjacency matrix where

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Also suppose that vertices  $v_1, v_2, v_3$  are to be assigned to colors  $c_1, c_2, c_3$  respectively. Assume  $n \geq n'$  where  $n' = 3$  in this case. Then

$$P_{8n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \begin{bmatrix} \\ \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} \\ \\ \end{bmatrix} \end{bmatrix}.$$

Then from Theorem 4.1,  $G(V, A, P_{8n})$  is  $n$ -colorable if and only if  $G'(V, A', \emptyset)$  is  $n$ -colorable where

$$A' = \begin{bmatrix} 0 & \underline{1} & 1 & 1 & 0 & 0 & 1 & 0 \\ \underline{1} & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$



according to the above definition of  $A'$ . Since  $V_1 = \{v_1\}$ ,  $V_2 = \{v_2\}$ , and  $V_3 = \{v_3\}$  then  $a'_{21}$ ,  $a'_{12}$ ,  $a'_{31}$ ,  $a'_{13}$ , and  $a'_{23}$  must equal 1 in  $A'$ . The underscored positions in  $A'$  indicate the elements that equal 0 in  $A$  and that have been reset to equal 1 in  $A'$ .

Thus the question of the existence of an  $n$ -coloration of a graph  $G$  with some vertices preassigned to specific colors has been reduced to the question of the existence of an  $n$ -coloration of a graph without any preassignment constraints.

The significance of Theorem 4.1 is that the existence of an  $n$ -coloration of graphs with preassignment constraints can now be determined using known methods that are feasible with respect to computation. That is, when  $n$  lies outside the interval  $[B_L, B_U)$  where  $B_L$  and  $B_U$  are as defined earlier or when the adjacency matrix of  $G'$  has the triangle property, or almost so, then the question of existence is answerable.

To illustrate the significance of Theorem 4.1 for the above example, suppose the question is to determine the existence of a 3-coloration of  $G(V, A, P_{83})$ . But  $G(V, A, P_{83})$  is 3-colorable if and only if  $G'(V, A', \emptyset)$  is 3-colorable. From matrix  $A'$ ,  $B_L = K(G') = 3$  and the best upper bound  $B_U$  obtained from previously known methods is  $B_U = 4$ . Hence 3 is in the interval  $[B_L, B_U)$ . Using the results presented in the previous chapter, it is easy to show that there does not exist a 3-coloration of  $G'(V, A', \emptyset)$ . Hence there does not exist a 3-coloration of  $G(V, A, P_{83})$ .



## 4.2 Prevention of Assignment Constraints

Another type of requirement that often arises in practise is that given vertices (classes) are not to be assigned to specified colors (hours). Such a requirement will be called a prevention of assignment constraint. The following shows how these constraints can be reduced to adjacency constraints.

Given a graph  $G(V, E)$  with  $|V| = m$  and a set  $S'$  of prevention of assignment constraints. The set of constraints  $S'$  can be expressed in a  $m \times n$  prevention of assignment matrix  $P'_{mn} = [p'_{ij}]$  where  $p'_{ij} = 1$  if vertex  $v_i \in V$  is not to be assigned to color  $c_j$  and  $p'_{ij} = 0$  otherwise.

A graph  $G(V, E)$  with preassignment matrix  $P_{mn}$  and prevention of assignment matrix  $P'_{mn}$  will be denoted by  $G(V, A, P_{mn}, P'_{mn})$ . If  $P_{mn} \equiv 0$ , denote the corresponding graph by  $G(V, A, \emptyset, P'_{mn})$ . Similarly for  $P'_{mn}$ .

Corresponding to  $G(V, A, \emptyset, P'_{mn})$ , define the graph  $\overline{G}(\overline{V}, \overline{A}, \overline{P}_{m+n \ n}, P'_{m+n \ n})$  with

1.  $\overline{V} = V \cup \{v_{m+1}, \dots, v_{m+n}\}$ ,
2.  $\overline{A} = [\bar{a}_{ij}]$  where  $\bar{a}_{ij} = 1$  if  $a_{ij} = 1$  and  $\bar{a}_{ij} = 0$  otherwise, and
3.  $\overline{P}_{m+n \ n} = [\bar{p}_{ij}]$  where  $\bar{p}_{m+i \ i} = 1$  for  $i = 1, \dots, n$ , and  $\bar{p}_{ij} = 0$

otherwise.

Then the following lemma gives the first of three equivalence statements.

**Lemma 4.2**  $G(V, A, \emptyset, P'_{mn})$  is  $n$ -colorable if and only if  $\overline{G}(\overline{V}, \overline{A}, \overline{P}_{m+n \ n}, P'_{m+n \ n})$

is  $n$ -colorable.



Proof: The proof is immediate since the  $n$  additional vertices  $v_{m+1}, \dots, v_{m+n}$  in the graph  $\overline{G}(\overline{V}, \overline{A}, \overline{P}_{m+n \ n}^i, P_{m+n \ n}^i)$  are 'independent' of the vertices in  $V$ . No vertex  $v_{m+i}$ ,  $i = 1, \dots, n$ , is adjacent to any vertex  $v_j \in V$ . Q.E.D.

Define the graph  $\overline{G}'(\overline{V}, \overline{A}', \emptyset, P_{m+n \ n}^i)$  corresponding to  $\overline{G}(\overline{V}, \overline{A}, \overline{P}_{m+n \ n}^i, P_{m+n \ n}^i)$  with  $\overline{A}' = [\bar{a}'_{ij}]$  where  $\bar{a}'_{ij} = 1$  if  $\bar{a}_{ij} = 1$  and  $\bar{a}'_{ij} = 1$  for  $i, j = m+1, \dots, m+n$ ,  $i \neq j$ . Otherwise  $\bar{a}_{ij} = 0$ .

The next lemma relates the coloration of  $\overline{G}(\overline{V}, \overline{A}, \overline{P}_{m+n \ n}^i, P_{m+n \ n}^i)$  and  $\overline{G}'(\overline{V}, \overline{A}', \emptyset, P_{m+n \ n}^i)$ .

Lemma 4.3  $\overline{G}(\overline{V}, \overline{A}, \overline{P}_{m+n \ n}^i, P_{m+n \ n}^i)$  is  $n$ -colorable if and only if

$\overline{G}'(\overline{V}, \overline{A}', \emptyset, P_{m+n \ n}^i)$  is  $n$ -colorable.

Proof:

$$\overline{A} = \begin{bmatrix} \begin{bmatrix} A \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} A_1 \end{bmatrix} \end{bmatrix} \text{ and } \overline{A}' = \begin{bmatrix} \begin{bmatrix} A \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} A_2 \end{bmatrix} \end{bmatrix} \text{ where}$$

the  $n \times n$  matrix  $A_1 \equiv 0$  and the  $n \times n$  matrix  $A_2$  has all nondiagonal elements equal to 1 and diagonal elements equal to 0.

Matrix  $P_{m+n \ n}^i$  applies only to vertices  $v_1, \dots, v_m \in V$ . Also matrices  $\overline{P}_{m+n \ n}^i$  and  $A_2$  express the same constraints. Thus the required result is proved. Q.E.D.



Finally, define the graph  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, \emptyset)$ , corresponding to the graph  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, P'_{m+n \ n})$ , with  $\bar{A}' = [\bar{a}'_{ij}]$  a symmetric matrix where  $\bar{a}'_{ij} = 1$  if  $\bar{a}'_{ij} = 1$  and for  $j = m + k$  if  $p'_{ik} = 1$ ,  $i = 1, \dots, m$ . Otherwise  $\bar{a}'_{ij} = 0$ .

Lemma 4.4  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, P'_{m+n \ n})$  is  $n$ -colorable if and only if  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, \emptyset)$  is  $n$ -colorable.

Proof: Suppose  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, P'_{m+n \ n})$  is  $n$ -colorable. Let  $\{C_k\}_{k=1}^n$  be an  $n$ -coloration of  $\bar{G}'$  where for each  $k$ ,  $k = 1, \dots, n$ , the set  $C_k$  contains the vertices of  $\bar{G}'$  that have been assigned to color  $c_k$ . By definition,  $C_k \cap C_{k'} = \emptyset$  for  $k \neq k'$ . To show that  $\{C_k\}_{k=1}^n$  is an  $n$ -coloration of  $\bar{G}'$ , it is required to show that for any  $v_i, v_{i'} \in C_k$ ,  $i \neq i'$  and  $1 \leq k \leq n$ , then  $\bar{a}'_{ij} = 0$ . Suppose to the contrary that  $\bar{a}'_{ij} = 1$ . Then by the definition of  $\bar{a}'_{ij}$  one of the following must hold.

1.  $\bar{a}_{ij} = \bar{a}'_{ij} = 1$ . But this contradicts  $\{C_k\}_{k=1}^n$  being an  $n$ -coloration of  $\bar{G}'$ .

2. Without loss of generality assume  $i > j$ . Then  $\bar{a}'_{ij} = 1$  implies that  $i = m + k$  and that  $p'_{jk} = 1$ . But this contradicts  $\{C_k\}_{k=1}^n$  being an  $n$ -coloration of  $\bar{G}'$  that satisfies the conditions in matrix  $P'_{m+n \ n}$ . Thus  $\{C_k\}_{k=1}^n$  must be an  $n$ -coloration of  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, \emptyset)$ .

To show the converse, let  $\{C_k\}_{k=1}^n$  be an  $n$ -coloration of  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, \emptyset)$  where the  $C_k$  are defined as above. None of the constraints of matrix  $\bar{A}'$  are violated in such a coloration, by definition of the matrix  $\bar{A}'$ . Both for  $\bar{G}'$  and  $\bar{G}'$ , each of the vertices  $v_{m+k}$ ,  $k = 1, \dots, n$ , belong to a different set  $C_{k'}$ ,  $k' = 1, \dots, n$ . This follows from the definition of matrices  $\bar{A}'$  and  $\bar{A}'$ . Suppose vertex  $v_{m+k} \in C_k$ ,  $k = 1, \dots, n$ . If there exists



$v_i \in C_k$  for some  $k$ ,  $k = 1, \dots, n$ , and  $i \leq m$  such that  $p_{ik} = 1$ , then

$\bar{a}_{i, m+k}'' = 1$ . But this is a contradiction. Hence  $\{C_k\}_{k=1}^n$  is an

$n$ -coloration of  $\bar{G}'$ . Q.E.D.

Lemmas 4.2, 4.3, and 4.4 provide the proof to the following theorem.

**Theorem 4.2** The graph  $G(V, A, \emptyset, P'_{m+n, n})$  is  $n$ -colorable if and only if the graph  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, \emptyset)$  is  $n$ -colorable.

**Proof:** From Lemma 4.2,  $G(V, A, \emptyset, P'_{mn})$  is  $n$ -colorable if and only if  $\bar{G}(\bar{V}, \bar{A}, \bar{P}_{m+n, n}, P'_{m+n, n})$  is  $n$ -colorable. Next from Lemma 4.3,  $\bar{G}(\bar{V}, \bar{A}, \bar{P}_{m+n, n}, P'_{m+n, n})$  is  $n$ -colorable if and only if  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, P'_{m+n, n})$  is  $n$ -colorable. Finally, from Lemma 4.4,  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, P'_{m+n, n})$  is  $n$ -colorable if and only if  $\bar{G}'(\bar{V}, \bar{A}', \emptyset, \emptyset)$  is  $n$ -colorable. Q.E.D.

To illustrate the result of Theorem 4.2, consider a graph  $G(V, A)$  with  $|V| = 6$  and an adjacency matrix  $A$  where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Furthermore, vertices  $v_1$ ,  $v_2$ , and  $v_3$  are not to be assigned to colors  $c_3$ ,  $c_2$ , and  $c_1$  respectively. Hence



$$P'_{6n} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad 0 \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

From Lemma 4.2,  $G(V, A, \emptyset, P'_{6n})$  is  $n$ -colorable if and only if

$\overline{G}(\overline{V}, \overline{A}, \overline{P}_{6+n \ n}, P'_{6+n \ n})$  is  $n$ -colorable where

$$\overline{V} = V \cup \{v_7, v_8, v_9, \dots, v_{6+n}\}$$

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad 0 \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

$$\overline{P}_{6+n \ n} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 0 \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$P'_{6+n \ n} = \begin{bmatrix} P'_{6n} \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 \end{bmatrix}.$$

From Lemma 4.3,  $\overline{G}(\overline{V}, \overline{A}, \overline{P}_{6+n \ n}, P'_{6+n \ n})$  is  $n$ -colorable if and only if

$\overline{G}'(\overline{V}, \overline{A}', \emptyset, P'_{6+n \ n})$  is  $n$ -colorable where

$$\overline{A}' = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & & 1 \\ & & \vdots & & \\ & 1 & & \vdots & \\ & \vdots & & 1 & \vdots \\ 1 & \dots & 1 & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix} \end{bmatrix}.$$

Finally from Lemma 4.4,  $\overline{G}'(\overline{V}, \overline{A}', \emptyset, P'_{6+n \ n})$  is  $n$ -colorable if and only if

$\overline{G}''(\overline{V}, \overline{A}'', \emptyset, \emptyset)$  is  $n$ -colorable where



$$A'' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ 1 \\ 1 \\ 0 \\ 1 \\ . \\ . \\ 1 \\ . \\ . \\ . \\ . \\ 1 \\ . \\ . \\ 1 \\ 0 \end{bmatrix} .$$

Thus the question of the existence of an  $n$ -coloration of a graph with vertices that are not to be assigned to specified colors has been reduced to the question of the existence of an  $n$ -coloration of a graph without those constraints. Furthermore, it is immediate that if  $P'_{mn}$  is such that there are really only  $n'$ ,  $n' < n$ , colors to which vertices are not to be assigned, then only  $n'$  'additional' vertices need to be introduced.

As for preassignment constraints, the significance of Theorem 4.2 is that the existence of an  $n$ -coloration with preassignment constraints can now be determined if  $n$  is not in the interval  $[B_L, B_U)$  where  $B_L$  and  $B_U$  are the best known bounds of  $\chi(\bar{G}')$  where  $\bar{G}' = \bar{G}'(V, A'', \emptyset, \emptyset)$  or when the adjacency matrix  $\bar{A}''$  has the triangle property, or almost so.

To illustrate the significance of Theorem 4.2 for the above example, suppose the question is to determine the existence of a



3-coloration of  $G(V, A, \emptyset, P_{63}^I)$ . But  $G(V, A, \emptyset, P_{63}^I)$  is 3-colorable if and only if  $\overline{G}^I(\overline{V}, \overline{A}^I, \emptyset, \emptyset)$  is 3-colorable. For  $n = 3$ , then

$$\overline{A}^I = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Using previously known methods for finding upper bounds  $B_U$  of  $\chi(\overline{G}^I)$ , it is immediate that there is an upper bound  $B_U = 3$ . Thus a 3-coloration of  $\overline{G}^I(\overline{V}, \overline{A}^I, \emptyset, \emptyset)$  exists and hence similarly for  $G(V, A, \emptyset, P_{63}^I)$ .

The statement of Theorem 4.2 applies to graphs  $G(V, A, P_{mn}, P_{mn}^I)$  with  $P_{mn} \neq 0$ . The extension to graphs  $G(V, A, P_{mn}, P_{mn}^I)$  with  $P_{mn} \neq 0$  and  $P_{mn}^I \neq 0$  is immediate. This is illustrated by the following example.

Consider a graph  $G(V, A)$  with  $|V| = 6$  and an adjacency matrix  $A$  where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$



Suppose vertices  $v_1$  and  $v_2$  are preassigned to color  $c_1$  and vertex  $v_5$  is not to be assigned to color  $c_3$ . That is, the preassignment matrix  $P_{63}$  and the prevention of assignment matrix  $P'_{6n}$  are

$$P_{6n} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & 0 & & 0 & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix} \quad \text{and} \quad P'_{6n} = \begin{bmatrix} 0 & 0 & 0 & & & \\ & 0 & 0 & 0 & & \\ & & 0 & 0 & 0 & \\ & & 0 & 0 & 0 & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 & 0 \end{bmatrix}.$$

Following the result in Lemma 4.2,  $G(V, A, P_{6n}, P'_{6n})$  is  $n$ -colorable (assume  $n \geq 3$  as otherwise no  $n$ -coloration exists) if and only if  $\overline{G}(\overline{V}, \overline{A}, \overline{P}_{7n}, P'_{7n})$  is  $n$ -colorable where

$$\overline{V} = V \cup \{v_7\},$$

$$\overline{A} = \begin{bmatrix} & & & & & & 0 \\ & & & & & & \vdots \\ & & & & & & \vdots \\ & & & & & & \vdots \\ & & & & & & 0 \\ 0 & - & - & - & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{P}_{7n} = \begin{bmatrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 0 & 0 & 1 & 0 & - & - & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 0 & & & & \\ & & 0 & & 0 & & \\ & & & & 0 & & \\ & & & & & & 0 \\ 0 & 0 & 1 & 0 & - & - & 0 \end{bmatrix}, \text{ and}$$



$$P_{7n}^i = \begin{bmatrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 0 & - & - & - & 0 \end{bmatrix} P_{6n}^i = \begin{bmatrix} 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & - & - & - & 0 \end{bmatrix} .$$

Next, following the result of Lemma 4.3,  $\overline{G}(\overline{V}, \overline{A}, \overline{P}_{7n}, P_{7n}^i)$  is  $n$ -colorable if and only if  $\overline{G}'(\overline{V}, \overline{A}', \emptyset, P_{7n}^i)$  is  $n$ -colorable where

$$\overline{A}' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Finally, following the result of Lemma 4.4,  $\overline{G}'(\overline{V}, \overline{A}', \emptyset, P_{7n}^i)$  is  $n$ -colorable if and only if  $\overline{G}''(\overline{V}, \overline{A}'', \emptyset, \emptyset)$  is  $n$ -colorable where

$$\overline{A}'' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} .$$



Thus the above graph has been reduced to a graph without preassignment and prevention of assignment constraints. Furthermore, this has been achieved by introducing only one additional vertex.

### 4.3 Determination of an n-Coloration

Known upper bounds  $B_U$  of  $\chi(G)$  usually provide a specific n-coloration for all  $n \geq B_U$ . The edges of a graph, including those which have been included in the graph so as to represent preassignment and prevention of assignment constraints, represent many of the requirements within practical CTT problems. However, in cases when there are other desirable requirements that any particular solution (coloration) should meet, then more flexible methods for obtaining a coloration are required. Assuming an n-coloration exists, some implications of Theorem 4.1 for a flexible method for determination of an n-coloration are given. This will be done in context of the multi-stage decision process. An important distinction should be noted. Previous discussion concerning the multi-stage decision process was in context of determination of the existence of an n-coloration whereas here it is in terms of finding an n-coloration which is assumed to exist.

Let  $G$  be graph with vertex set  $V$ ,  $|V| = m$ , edge set  $E$ , and adjacency matrix  $A$ . Consider the problem of finding an n-coloration of  $G$  as a m-stage decision problem where at each stage  $i$ ,  $1 \leq i \leq m$ , a decision pertaining to assigning  $v_i \in V$  to one of the  $n$  colors  $\{c_j\}_{j=1}^n$  must be made as discussed in Chapter 2. That is, vertex  $v_i$  must not be assigned to any color  $c_j$ ,  $1 \leq j \leq n$ , such that there exists  $v_k \in V$ ,  $k < i$ , with edge  $(v_i, v_k) \in E$  and  $v_k$  having been assigned to



color  $c_j$ . Also it is desirable that the decision be made so that each of the vertices  $v_k$ ,  $k > i$ , can be assigned to some color  $c_j$ ,  $j = 1, \dots, n$ , without having to revise previous decisions to prevent assigning adjacent vertices to the same color.

Suppose stages  $1, \dots, i - 1$  have been completed and vertex  $v_i$  must be assigned to a color according to the above two specifications. Methods to achieve the first are straightforward. Thus assume that  $v_i$  can be assigned to each of the colors  $c_j$ ,  $j \in J_i$  where

$J_i = \{k \mid 1 \leq k \leq n \text{ and } k \text{ such that there does not exist a vertex } v_q, q < i, \text{ for which edge } (v_i, v_q) \in E \text{ of } G \text{ and } v_q \text{ having been assigned to color } c_k\}$ .

In general the problem is to determine a set  $J_i^!$  where

$$J_i^! = \{k \mid k \in J_i \text{ and if vertex } v_i \text{ is assigned to color } c_k$$

then vertices  $v_q$ ,  $q > i$ , can be assigned to one of the  $n$  colors available\}.

Suppose a decision at some stage  $k$  has been completed in that vertex  $v_k$  has been assigned to a specific color. This is equivalent to the vertex having been assigned to the particular color. Thus while a decision is being made at stage  $i$ , those made at stages  $1, \dots, i - 1$  are equivalent to vertices  $v_1, \dots, v_{i-1}$  having been preassigned to specific colors. Furthermore, to determine  $J_i^! \subseteq J_i$ , consider  $v_i$  temporarily assigned to some color  $c_j$ ,  $j \in J_i$ . Then considering vertices  $v_1, \dots, v_i$  as having been preassigned to specific colors,



Theorem 4.1 may be used to determine if  $j \in J_i^!$ . Thus all known methods related to establishing the existence of an  $n$ -coloration of a graph can be used to determine  $J_i^!$  for each  $i$  and hence a coloration of a graph.

To illustrate the effectiveness of the above considerations, consider a graph  $G$  with vertices  $v_1, \dots, v_7$  and adjacency matrix  $A$  where

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Determine a 4-coloration of  $G$  by means of a 7-stage decision process.

Suppose stage 3 has been completed with  $v_1$  having been assigned to color  $c_1$  and  $v_2$  and  $v_3$  having been assigned to color  $c_2$ . Using Theorem 4.1, graph  $G$  with stages 1, 2, and 3 completed can be represented by a graph  $G'$  with vertices  $v_1, \dots, v_7$  and adjacency matrix  $A'$  where

$$A' = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$



But from  $A'$ , then  $k(G') = 5$ . It is immediate that vertices  $v_2$  and  $v_3$  cannot be assigned to the same color. This avoids the problem of continuing to stages 4, 5, etc., only to find that back-up to stage 3 is required so as to reassign vertex  $v_3$ .

The above should not imply the need for necessarily determining  $J_i^!$  or any subset thereof at each stage  $i$  in order to use the previous results. For example, if  $n \gg B_U$  then it could be advantageous to assign  $v_i$  to color  $c_k$  for some  $k \in J_i$  and continue to stage  $i + 1$  without determining  $J_i^!$ . The need for determining  $k$ ,  $k \leq i$ , such that  $J_k^! \neq \emptyset$  and  $J_{k+1}^! = \emptyset$  now arises only when  $J_{i+1} = \emptyset$ . If  $n \gg B_U$  this could be expected to occur frequently.



## CHAPTER 5

### RESULTS AND CONCLUSIONS

The following CTT problem illustrates the correspondence between a CTT problem and the graph models discussed. The problem also demonstrates the application of the results given in the previous chapters.

Consider a CTT problem with 12 classes, each of which are to meet for one hour, beginning on the hour, every morning between 8:00 A.M. and 12:00 noon. Suppose the classes have been denoted by  $v_1, \dots, v_{12}$ . The classes are to be assigned or scheduled to one of the four available hours according to the following requirements:

1. Teachers  $t_1, t_2, t_3, t_4, t_5$  and  $t_6$  are to meet classes  $\{v_2, v_7, v_8\}$ ,  $\{v_{10}, v_{12}\}$ ,  $\{v_3, v_6\}$ ,  $\{v_9, v_{11}\}$ ,  $\{v_1, v_5\}$ , and  $\{v_4\}$  respectively.

2. The following pairs of classes correspond to courses within academic programs, and hence must not conflict; that is, the classes must not meet during the same hour:  $(v_2, v_1)$ ,  $(v_3, v_2)$ ,  $(v_4, v_2)$ ,  $(v_4, v_3)$ ,  $(v_5, v_2)$ ,  $(v_5, v_3)$ ,  $(v_7, v_5)$ ,  $(v_7, v_6)$ ,  $(v_9, v_3)$ ,  $(v_9, v_6)$ ,  $(v_{10}, v_2)$ ,  $(v_{10}, v_6)$ ,  $(v_{11}, v_6)$  and  $(v_{12}, v_6)$ .



3. The following pairs of classes are popular among students in the sense that many enroll in both of them:  $(v_3, v_1)$ ,  $(v_8, v_5)$ ,  $(v_9, v_8)$  and  $(v_{12}, v_2)$ .

4. Teachers  $t_4$  and  $t_6$  are in charge of student counselling, and one of them must be available at all times.

5. Teacher  $t_3$  must meet his class  $(v_6)$  at 8:00 A.M.

6. Teacher  $t_4$  is not available for teaching at 9:00 A.M.

7. Class  $v_{10}$  must meet in a particular classroom. This classroom is not available to class  $v_{10}$  at 11:00 A.M.

8. Classes  $v_2, v_3, v_4, v_5$  are senior classes and are not to meet prior to 9:00 A.M.

9. Several of the students on Student Council, which meets at 9:00 A.M. every day, have enrolled (plan to enroll) in class  $v_{12}$ .

10. Some of the students who have enrolled (plan to enroll) in  $v_{11}$  are members of an athletic team that practises at 9:00 A.M.

Let the classes  $v_i$  correspond to the vertices  $V = \{v_i\}$  of a



graph  $G$  with  $V = 12$ . Define the non-zero elements below the main diagonal of an adjacency matrix  $A$  of  $G$  as follows:

1. Corresponding to the first of the above requirements, consider classes  $v_2$ ,  $v_7$  and  $v_8$  which must be met by teacher  $t_1$ . Teacher  $t_1$  is prevented from having to meet more than one class during any given hour by setting  $a_{72}$ ,  $a_{82}$ ,  $a_{87}$  equal to 1; similarly, for the other classes met by the same teacher.
2. Corresponding to the second requirement, set  $a_{21}$ ,  $a_{32}, \dots$ ,  $a_{12,6}$  equal to 1, so as to prevent conflicts from occurring within any solution.
3. Corresponding to the third requirement, set  $a_{31}$ ,  $a_{85}$ ,  $a_{98}$  and  $a_{12,2}$  equal to 1, so as to allow students to enroll in these popular combinations.
4. Corresponding to the fourth requirement, set  $a_{94}$ ,  $a_{11,4}$ , and  $a_{11,9}$  equal to 1. This prevents both teachers  $t_4$  and  $t_6$  from meeting their classes during the same hour.

Since  $A$  is symmetric,  $A$  is defined completely. Thus,



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The fifth requirement corresponds to a preassignment constraint where class  $v_6$  must meet at 8:00 A.M.. Thus the preassignment matrix  $P_{12,4}$  is defined as



$$P_{12,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the 4 columns of  $P_{12,4}$  correspond to the 4 available hours and colors. Colors  $c_1$ ,  $c_2$ ,  $c_3$  and  $c_4$  correspond to 8:00 A.M., 9:00 A.M., 10:00 A.M. and 11:00 A.M. respectively.

Requirements 6 through 10 correspond to prevention of assignment constraints. Note that requirement 10 is redundant following the requirement 6. Thus the prevention of assignment matrix  $P'_{12,4}$  is defined as



$$P'_{12,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The question concerning the existence of a solution to the CTT problem is equivalent to the question as to the existence of a 4-coloration for the graph  $G(V, A, P_{12,4}, P'_{12,4})$ . Following the results and illustrations in Chapter 4,  $G(V, A, P_{12,4}, P'_{12,4})$  is 4-colorable if and only if  $G'(\bar{V}, \bar{A}', \emptyset, \emptyset)$  is 4-colorable where  $V = V \cup \{v_{13}, v_{14}, v_{15}\}$ , and



$$\bar{A}'' = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} v_1 \\ \\ \\ \\ \\ v_{12} \\ \\ v_{15} \end{matrix}$$

Following the example given in Chapter 3, the upper bounds of  $\chi(\bar{G}'')$  of Brooks [11], Welsh and Powell [46], Peck and Williams [41], Williams [47], and Formby [20] are 12, 6, 5, 6, and 5, respectively. Thus 4 lies in the interval  $[B_L, B_U) = [4, 5)$ ; hence, the existence of a 4-coloration of  $\bar{G}''$  cannot be determined from the known bounds of  $\chi(\bar{G}'')$ .



The submatrices indicated in  $\bar{A}''$  show that  $\bar{A}''$  has the form corresponding to the almost algebraic cases discussed in Section 3.3.4. Hence, the computation of  $|U_{(ij) \in I} P_{ij}|$  is immediate, following the example given in Chapter 3, and is found to equal 24. Thus  $\bar{G}''$  has a 4-coloration, and hence, similarly for  $G$ . Therefore, the above CTT problem does have a solution.

To find a 4-coloration of  $\bar{G}''$ , let  $c_1, c_2, c_3$  and  $c_4$  be the 4 available colors and use a 15-stage decision process to find a 4-coloration, which is known to exist, as discussed in Section 4.3. Let the  $i^{\text{th}}$  stage correspond to assigning vertex  $v_i$  to some color  $c_j$ . Then the first three stages allow no choice. After the completion of each stage, it is immediate that no incorrect decision is possible as  $\bar{G}''(\bar{V}, \bar{A}'', \emptyset, \emptyset)$  with the corresponding preassignment constraints is equivalent to  $\bar{G}''(\bar{V}, \bar{A}'', \emptyset, \emptyset)$  without the corresponding preassignment constraints, from Theorem 4.1. Hence assign  $v_1$  to  $c_1$ ,  $v_2$  to  $c_2$  and  $v_3$  to  $c_3$ . At stage 4, temporarily assign vertex  $v_4$  to  $c_1$ . From  $\bar{A}''$ , such an assignment is valid; and hence, 1 is an element of  $J_4$ , where  $J_4$  is defined as in Section 4.3. Consider now the graph  $\bar{G}''(\bar{V}, \bar{A}'', \emptyset, \emptyset) = \bar{\bar{G}}(\bar{V}, \bar{\bar{A}}'', \emptyset, \emptyset)$  with the 4 preassignment constraints corresponding to the first 4 stages and where  $\bar{\bar{A}}'' = \bar{A}''$ . From Theorem 4.1, such a graph is 4-colorable if and only if the graph  $\bar{\bar{G}}'(\bar{V}, \bar{\bar{A}}'', \emptyset, \emptyset)$  is 4-colorable where



$$\overline{\overline{A'}} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

But for  $\overline{\overline{G'}}(\overline{\overline{V}}, \overline{\overline{A'}}, \emptyset, \emptyset)$ ,  $K(\overline{\overline{G'}}) = 5$ . Hence  $\overline{\overline{G'}}$  is not 4-colorable and 1 is not an element of  $J_4^1$  where  $J_4^1$  is defined as in Section 4.3. Hence 4 is in  $J_4^1$ , as 2 and 3 are not in  $J_4$ , and  $\overline{\overline{G}}$  is known to have a 4-coloration. Thus assign  $v_4$  to  $c_4$ . Continuing in this manner, the following 4-coloration is obtained:  $\{v_1, v_6, v_8\}$ ,  $\{v_2, v_9, v_{14}\}$ ,  $\{v_3, v_7, v_{11}, v_{12}, v_{15}\}$ ,  $\{v_4, v_5, v_{10}, v_{13}\}$  are assigned to colors  $c_1, c_2, c_3$  and  $c_4$  respectively. Now,  $\overline{\overline{A'}}$  was defined so as to correspond  $v_6, v_{13}, v_{14}$  and  $v_{15}$  to classes that are to meet at 8:00 A.M., 9:00 A.M.,



10:00 A.M., and 11:00 A.M. respectively. Then a solution to the CTT problem has been obtained by associating  $c_1$ ,  $c_4$ ,  $c_2$  and  $c_3$  with 8:00 A.M., 9:00 A.M., 10:00 A.M., and 11:00 A.M. respectively. Thus the classes that meet at 8:00 A.M., 9:00 A.M., 10:00 A.M. and 11:00 A.M. are  $\{v_1, v_6\}$ ,  $\{v_4, v_5, v_{10}\}$ ,  $\{v_2, v_9\}$ , and  $\{v_3, v_7, v_{11}, v_{12}\}$  respectively. It is easy to check that all the given requirements have been satisfied.

For the sake of clarity, define the following four sets of graphs, all of which may correspond to CTT problems: Let  $\text{Class}_1$  be the set of all graphs;  $\text{Class}_2$  be the set of all graphs with preassignment constraints;  $\text{Class}_3$  be the set of all graphs with prevention of assignment constraints; and,  $\text{Class}_4$  be the set of all graphs with preassignment and prevention of assignment constraints.

The number of cases for which the determination of the existence of a solution can be realized has been improved for CTT problems. The improvement is based on identifying a property, called the triangle property, of the adjacency matrix of a graph.

Theorem 4.1 shows that for any CTT problem that is equivalent to a graph in  $\text{Class}_2$  for which an  $n$ -coloration is to be obtained, is also equivalent to a graph in  $\text{Class}_1$  for which an  $n$ -coloration is to be obtained. Similarly, Theorem 4.2 shows that for any CTT problem that



is equivalent to a graph in  $\text{Class}_3$  for which an  $n$ -coloration is to be obtained, is also equivalent to a graph in  $\text{Class}_1$  for which an  $n$ -coloration is to be obtained. There are many graphs in  $\text{Class}_1$  for which known methods are able to determine the existence of an  $n$ -coloration. Hence there are many CTT problems, previously unanswered, for which the given results enable the determination of the existence of a solution. The same conclusion also applies to CTT problems that are equivalent to graphs in  $\text{Class}_4$ .

A correspondence between the completed stages of a multi-stage decision process and preassignment constraints was shown in Section 4.3. Gotlieb [21] gave a necessary condition for the existence of a solution to CTT(A) problems with preassignment constraints. He stated that it must hold after stage  $i$  in order to possibly attain a solution. However, as pointed out earlier, Gotlieb's condition is not sufficient; plus, it is restricted to a particular class of problems. The condition given in Section 4.3, the existence of a coloration, is both necessary and sufficient. It provides the basis for a very flexible means of determining an  $n$ -coloration of a graph in either of  $\text{Class}_i$ ,  $i = 1, 2, 3$  or  $4$ . More importantly, the same applies for determining a solution of a CTT problem that is equivalent to such graphs.



For the CTT problems considered by Gotlieb (previously labelled as CTT(A) problems), no condition which is both necessary and sufficient for the existence of a solution was known when preassignment constraints are present, except for problems with only one class or one teacher. From the concept of introducing new edges as for graphs in Class<sub>i</sub>,  $i = 1, 2, 3$  and 4, such a necessary and sufficient condition is immediate.

A CTT(A) problem with no preassignment constraints has a solution, provided no teacher or no class is involved in more than  $\gamma$  meetings. Given a CTT(A) problem that has a solution and no preassignment constraints, let the  $\gamma$  meetings of a class  $c_j$  be represented by  $\gamma$  distinct vertices. Doing this for each class, a total of  $\beta \cdot \gamma$  vertices is obtained. Subdivide each set of  $\gamma$  vertices corresponding to a class into subsets such that the vertices of any such subset corresponds to meetings with a particular teacher. If  $\sum_i r_{ij} < \gamma$ , then there will be meetings for which class  $c_j$  does not meet with any teacher. Let the corresponding vertices belong to a subset which corresponds to meeting with a dummy teacher. Join every pair of vertices in each subset for each class, including the one corresponding to meeting the dummy teacher. Next join each pair of vertices belonging to subsets which correspond to different classes and which meet the same teacher, excluding those meeting a dummy teacher. Then, a graph  $G$  corresponding to the CTT(A) problem with no preassignment constraints is obtained.



Using the results presented, a graph  $G'$  is obtained that corresponds to the CTT(A) problem with preassignment constraints. Then a necessary and sufficient condition for the existence of a solution to the corresponding CTT(A) problem is that  $G'$  have a  $\gamma$ -coloration; similarly, for CTT(A) problems with prevention of assignment constraints.

Theorem 3.10 shows, by means of identifying properties of a sequence of adjacency matrices, that the four color problem is equivalent to showing the medial graph of any cubic graph to be a perfect graph. The theorem is of interest from the point of recognizing a difficult and unsolved problem from yet another direction.

Finally, the relationship which is shown to exist between the triangle property and triangulated graphs hints at relationships between the problems being faced by graph theorists and computer scientists. These problems are the characterization of graphs and the search for efficient algorithms for certain combinatorial problems. These problems appear to be perpetually unsolvable.

The advances reported are relevant to other resource allocation problems. For example, the results apply to job scheduling problems. As shown by Welsh and Powell [46], each job scheduling problem



is a graph  $G(V,E)$  with  $|V| = m$  where  $m$  jobs are to be scheduled during different days such that certain pairs of jobs are not scheduled on the same day. The solution seeks the minimum number of days required for such a schedule. Finding an upper bound of the chromatic number of a graph corresponds to finding a sub-optimal solution to the above job scheduling problem. The results presented permit finding sub-optimal solutions to such job scheduling problems, with the additional constraints that some jobs have been prescheduled to specific days, and/or some jobs are not to be scheduled on specified days.

This study lends itself to some further research of an experimental nature. First, the results reported allow the comparison of the chromatic number  $\chi(G)$  to known upper bounds of  $\chi(G)$  for graphs  $G$  corresponding to adjacency matrices that have the triangle property, or almost so. Welsh and Powell [46] suggested such a comparison. Previously, such a comparison was feasible only for graphs that were known to be perfect, that is  $\chi(G) = K(G)$ . This should be done so as to learn as much as possible concerning the chromatic number and known upper bounds of the chromatic number; this applies particularly to cases when known methods are unable to determine the chromatic number of a graph. Second, future work should relate graph models to CTT(B) problems with courses that are offered more than once (multi-sectioned courses).



## BIBLIOGRAPHY

1. Akers, Jr., S.E., and Friedman, J., 'A Non-Numerical Approach to Production Scheduling Problems', *Operations Research*, Vol. 3, 1955, pp. 429-442.
2. Almond, M., 'An Algorithm for Constructing University Timetables', *Computer Journal*, Vol. 8, 1965-66, pp. 331-340.
3. Appleby, J.S., Blake, D.V., and Newman, E.A., 'Techniques for Producing School Timetables on a Computer and Their Application to Other Scheduling Problems', *Computer Journal*, Vol. 3, 1961, pp. 237-244.
4. Balinski, Michel L., and Quandt, R.E., 'On an Integer Program for a Delivery Program', *Operations Research*, Vol. 12, 1964, pp. 300-304.
5. Balinski, Michel L., 'Integer Programming: Methods, Uses, Computation', *Management Science*, Vol. 12, No. 3, November 1965, pp. 253-313.
6. Barraclough, Elizabeth, 'The application of a Digital Computer to the Construction of Timetables', *Computer Journal*, Vol. 8, 1965-66, pp. 136-146.
7. Berge, C., The Theory of Graphs and its Applications, Methuen & Co. Ltd., London.
8. Berge, C., 'Some Classes of Perfect Graphs', Graph Theory and Theoretical Physics (Edited by F. Harary), 1967, Academic Press, pp. 155-165.
9. Bonzon, P., 'Necessary and Sufficient Conditions for Dynamic Programming of Combinatorial Type', *J. ACM*, Vol. 17, No. 4, October 1970, pp. 675-682.
10. Broder, S., 'Final Exam Scheduling', *Comm. ACM*, Vol. 7, No. 8, August 1964, pp. 494-498.
11. Brooks, R.L., 'On Coloring the Nodes of a Network', *Proc. Cambridge Phil. Soc.*, Vol. 37, 1941, pp. 194-197.
12. Cole, A.J., 'The Preparation of Examination Timetables using a Small-store Computer', *Computer Journal*, Vol. 7, 1964-65, pp. 117-121.
13. Csima, J. and Gotlieb, C.C., 'A Computer Method For Constructing School Time-Tables', *Proceedings of the Eighteenth Annual Conference of the ACM*, August 1963.



14. Dantzig, G.B., 'On The Significance of Solving Linear Programming Problems With Some Integer Variables', *Econometrica*, Vol. 28, 1960, pp. 30-44.
15. Dempster, M.A.H., 'On the Gottlieb-Csima Time-tabling Algorithm', *Canadian Journal of Mathematics*, Vol. 20, pp. 103-119.
16. Dempster, M.A.H., 'Two Algorithms for the Time-table Problem', Combinatorial Mathematics And Its Applications (Edited by D.J.A. Welsh), July 1969, Academic Press Inc. (London) Ltd, pp. 63-85.
17. De Werra, D., 'Construction of School Timetables By Flow Methods', *Infor*, Vol 1, No. 1, pp. 12-22.
18. Edmonds, J., 'Paths, Trees, and Flowers', *Canadian Journal of Mathematics*, Vol. 17, 1965, pp. 449-467.
19. Edmonds, J., 'Maximum Matching and a Polyhedron with 0, 1-Vertices', *J. Research National Bureau of Standards*, B, 1965, pp. 125-130.
20. Formby, J.A., 'Computer Procedure For Bounding The Chromatic Number Of A Graph', Combinatorial Mathematics And Its Applications (Edited by D.J.A. Welsh), July 1969, Academic Press Inc. (London) Ltd, pp. 111-114.
21. Gottlieb, C.C., 'The Construction Of Class-Teacher Time-Tables', *Proceedings of IFIP Congress 62*, Munich, North Holland Pub. Co., Amsterdam, 1963, pp. 73-77.
22. Graves, G., and Whinston, A., 'A New Approach To Discrete Mathematical Programming', *Management Science*, Vol. 15, No. 3, 1968, pp. 177-190.
23. Greko, B., 'School Scheduling Through Capacitated Network Flow Analysis', *Planeringsgruppen*, Karlavagan 96, Stockholm, Sweden, September 1965.
24. Hall, A.D., and Forman, S., 'Scheduling University Course Examinations By Computer', *Comm. ACM*, Vol. 10, No. 4, April 1967, pp. 235-238.
25. Hall, M., 'An Algorithm for Distinct Representatives', *Amer. Math. Monthly*, Vol. 63, December 1956, pp. 716-717.
26. Hammer, Peter L., and Rudeanu, Sergiu, Boolean Methods in Operations Research, Springer Verlag New York Inc. 1968.
27. Harary, F., A Seminar on Graph Theory, Holt, Rinehart and Whinston, Inc., 1967.



28. Holzman, A.G., Optimal Scheduling in Educational Institutions, Cooperative Research Project No. 1323, University of Pittsburgh, Pittsburgh, Penn., 1964.
29. Hu, T.C., Integer Programming and Network Flows, Addison-Wesley Publishing Company, 1969.
30. Karp, R.M., 'Reducibility Among Combinatorial Problems', Technical Report, April 1972, Department of Computer Science, University of California (Berkeley).
31. Kehl, W.B., 360 GASP III Generalized Academic Simulation Programs, IBM Program Documentation.
32. Lions, J., 'Matrix Reduction Using the Hungarian Method for the Generation of School Timetables', Comm. ACM, Vol. 9, No. 5, May 1966, pp. 349-354.
33. Lions, J., 'A Counter-Example for Gotlieb's Method for the Construction of School Timetables', Comm. ACM, Vol. 9, No. 9, September 1966, pp. 697-698.
34. Lions, J., 'The Ontario School Scheduling Program', Computer Journal, Vol. 10, 1967-68, pp. 14-21.
35. Lions, J., 'A Generalization Of A Method For The Construction Of Class/Teacher Timetables', Information Processing 68, Proceedings of IFIP Congress 1968, North Holland Publishing Company, Amsterdam, pp. 1377-1382.
36. Lions, J., 'Some Results Concerning the Reduction of Binary Matrices', J. ACM, Vol. 18, No. 3, July 1971, pp. 424-430.
37. Myers, B.R., and Liu, R., 'A Lower Bound on the Chromatic Number of a Graph', Networks, Vol. 1, No. 3, 1972, pp. 273-277.
38. Nordhaus, E.A., 'On the Density and Chromatic Number of Graphs', The Many Facets of Graph Theory, Conference Proceedings, Western Michigan University, Kalamazoo, Michigan, October 1968, pp. 245-249.
39. Oakford, R.V., Coombs, A.F., Chatterton, L.A., and Kessler, R., Stanford School Scheduling System, Data Processor's Manual, School of Education and Department of Industrial Engineering, Stanford University, Stanford, California.
40. Ore, Oystein, The Four Color Problem, Academic Press, 1967.
41. Peck, J.E.L., and Williams, M.R., 'Algorithm 286, Exam Scheduling', Comm. ACM, Vol. 9, No. 6, June 1966, pp. 433-434.



42. Reiter, S., and Sherman, G.R., 'Discrete Optimizing', Journal of the Society for Industrial and Applied Mathematics, Vol. 13, No. 3, 1965, pp. 864-889.
43. Sherman, Gordon R., 'A Combinatorial Problem Arising From Scheduling University Classes', Journal of the Tennessee Academy of Science, Vol. 38, No. 3, July 1963, pp. 115-117.
44. Turksen, I.B., and Holzman, A.G., 'Feasible Sets For A Class Of Scheduling Problems', Session 70, The Information Processing Society of Canada and Canadian Operational Research Society, June 1970, Vancouver, Canada.
45. Waterloo Academic Scheduling System (WASS) Handbook, Office of the Registrar, University of Waterloo.
46. Welsh, D.J.A., and Powell, M.B., 'An Upper Bound for the Chromatic Number of a Graph and its Application to Timetabling problems', Computer Journal, Vol. 10, 1967-68, pp. 85-86.
47. Williams, M.R., 'The Coloring Of Very Large Graphs', Combinatorial Structures And Their Application, Proceedings of the Calgary International Conference on Combinatorial Structures and Their Application, June 1969, Calgary, Canada, Gordon and Breach, Science Publishers, pp. 477-478.
48. Winters, W.K., 'An Investigation Of Several Algorithms For Solving The Timetable Problem', CARS Report #6, July 1968, The University of Tennessee, Knoxville, Tennessee.
49. Witzhall, C., and Zahn, Jr., C.T., 'Modification of Edmond's Maximum Matching Algorithm', J. Research National Bureau of Standards, B, Mathematics and Mathematical Physics, 69B, 1965, pp. 91-98.
50. Zehnder, C.A., Mathematische Untersuchungen Zur Berechnung Von Stundenplänen Und Transportfahrplänen, Ph.D. Thesis, Eidgenössischem Technischen Hochschule, Zurich, 1965.

















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